Impact Factor: 9.9 ISSN-L: 2544-980X

Vaqt Yo'nalishlari Turlicha Bo'lgan Parabolo-Giperbolik Tenglama Uchun Birinchi Tur Integral Shartli Masala

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Annotatsiya: Mazkur ishda vaqt yo'nalishlari turlicha bo'lgan aralash tipdagi bir tenglama uchun qo'yilgan birinchi tur integral shartli masala yechimining mavjudligi va yagonalini isbotlangan.

Kalit so'zlar: Aralash tipdagi tenglama, birinchi tur integral shartli masala, integral tenglamalar usuli.

D orqali ushbu sohani belgilaylik, $D = D_0 \cup D_1 \cup D_2$ bu yerda

$D_0 = \{(x,t): 0 < x < +\infty,\ 0 < t < 1\}$,

$D_1 = \{(x,t): -x < t < x-1,\ 0 < x < 1/2\}$, $\quad D_2 = \{(x,t): -1 < x < 0,\ 0 < t < 1\}$.

Ushbu $I_1 = \{(x,t): -1 < x < 0,\ t = 0\}$, $I_2 = \{(x,t): 0 < t < 1,\ x = 0\}$ belgilashlarni kiritib, quyidagi (1) tenglamani qaraymiz:

$$0 = Lu \equiv \begin{cases} u_{xx} - u_t = 0, & (x,t) \in D_0, \\ u_{xx} - u_{tt} = 0, & (x,t) \in D_1, \\ u_{tt} + u_x = 0, & (x,t) \in D_2 \end{cases} \quad (1)$$

va (1) tenglama uchun D sohada ushbu masalani o'rganamiz:

1-masala. $Lu = 0$ tenglamaning

$$u(x,0) = \varphi_1(x), \ \ 1 \le x < +\infty; \quad (2)$$

$$u(x,0) = \varphi_2(x), \ \ -1 \le x \le 0; \quad (3)$$

$$\begin{aligned} &\lim_{x \to -0} u_x(x,t) = a_1(t) \lim_{x \to +0} u_x(x,t) + a_2(t) D_{0t}^{\alpha}[b_2(t)u(0,t)] + \\ &+ \alpha_3(t) D_{t1}^{\beta}[b_3(t)u(0,t)] + b_1(t), \quad 0 < t < 1; \end{aligned} \quad (4)$$

$$\lim_{x \to +\infty} u(x,t) = 0, \quad 0 \le t \le 1; \quad (5)$$

$$\int_0^1 u(x,t)\,dt = \varphi_3(x), \qquad -1 \le x \le 0 \quad (6)$$

$$u(x,-x) = \psi_1(x), \ \ 0 \le x \le 1/2; \quad (7)$$

$$u(x,x-1) = \psi_2(x), \ \ 1/2 \le x \le 1 \quad (8)$$

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shartlarni qanoatlantiruvchi $u(x,t)$ regulyar yechimi topilsin, bu yerda $a_j(t)$, $b_j(t)$, $\varphi_j(x)$, $j=\overline{1,3}$ va $\psi_i(x)$, $i=\overline{1,2}$ berilgan uzluksiz funksiyalar bo'lib,

$$a_j(t)\in C^1[0,1],\ a_1(t)\neq 0,\ \ t\in[0,1];\ \ b_j(t)\in C[0,1];\ \ \varphi_1(0)=\varphi_2(0)=0,\ \ \varphi_1(x)\in C[0,\infty),$$

$$\exists\varphi_1'(x)\in C(0,+\infty)\cap L(0,+\infty) \text{ va } \lim_{x\to+\infty}\varphi_1(x)=0;\ \ \varphi_2(x),\ \varphi_3(x)\in C^1[-1,0];$$

α va β lar esa $(0,1)$ segmentga tegishli bo'lgan berilgan haqiqiy sonlar; D_{0t}^{α} va D_{t1}^{β} lar kasr tartibli differensial operatorlar.

Masalaning tadqiqoti. Qo'yilgan masalaning bir qiymatli yechilishini isbotlaymiz. Faraz qilaylik, $u(x,t)$ funksiya masalaning yechimi bo'lsin. Masala shartlariga asoslanib, quyidagi belgilashlarni kiritaylik:

$$u(x,-0)=u(x,+0)=\tau_1(x),\quad 0\le x\le 1;\ (9)$$

$$u(-0,t)=u(+0,t)=\tau_2(t),\quad 0\le t\le 1;\ (10)$$

$$\lim_{x\to+0}u_x(x,t)=\tau_3(t),\quad 0<t<1,\ (11)$$

bu yerda $\tau_1(x)$, $\tau_2(t)$, $\tau_3(t)$ - noma'lum funksiyalar.

U holda masala yechimini D_1 sohada $u_{xx}-u_{tt}=0$ tenglama uchun (6) va $u(x,+0)=\tau_1(x),\quad x\in[0;1]$ shartlarni qanoatlantiruvchi yechimi sifatida ushbu

$$u(x,t)=\tau_1(x+t)+\psi_1\left(\frac{1+x-t}{2}\right)-\psi_1\left(\frac{1+x+t}{2}\right) \qquad (12)$$

ko'rinishda aniqlashimiz mumkin [1]. (12) yechimni (8) shartga bo'ysundirib, noma'lum $\tau_1(x)$ funksiyani ushbu ko'rinishda topamiz:

$$\tau_1(x)=\psi_1\left(\frac{x+1}{2}\right)+\psi_2\left(\frac{x+1}{2}\right)-\psi_1\left(\frac{1}{2}\right),\quad 0\le x\le 1.\ (13)$$

Endi masalani D_0 sohada qaraylik. Ma'lumki, $u_{xx}-u_t=0$ tenglamaning D_0 sohaning yopig'ida aniqlangan, uzluksiz hamda (2), (5) va ushbu

$$u(x,0)=\tau_1(x),\quad 0\le x\le 1;\ (14)$$

$$\lim_{x\to+0}u_x(x,t)=\tau_3(t),\quad 0<t<1;\ (15)$$

shartlarni qanoatlantiruvchi yechimi ushbu

$$u(x,t)=\int_0^1\frac{\sqrt{x\xi}}{2t}I_{-1/2}\left(\frac{x\xi}{2t}\right)e^{-(x^2+\xi^2)/4t}\tau_1(\xi)d\xi+$$
$$+\int_1^{+\infty}\frac{\sqrt{x\xi}}{2t}I_{-1/2}\left(\frac{x\xi}{2t}\right)e^{-(x^2+\xi^2)/4t}\varphi_1(\xi)d\xi-\frac{1}{\sqrt{\pi}}\int_0^t\tau_3(\eta)(t-\eta)^{-1/2}e^{-x^2/4(t-\eta)}d\eta \qquad (16)$$

ko'rinishda aniqlanadi [2], bu yerda $I_p(z)-$mavhum argumentli Bessel funksiyasi bo'lib [3,4], quyidagi ko'rinishda aniqlanadi:

$$I_p(z)=\sum_{k=0}^{+\infty}\frac{(z/2)^{2k-1/2}}{k!\Gamma(k+1/2)}.$$

Endi qaralayotgan tenglama va masalaning (2), (3), (6) shartlaridan foydalanib,

$$\tau_3(t)=-\frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t(t-\eta)^{-1/2}\left[\tau_2(\eta)-\Phi_1(\eta)\right]d\eta,\ t\in(0,1) \quad (17)$$

va quyidagi

$$\left.\begin{aligned}&\tau_2''+\lim_{x\to-0}u_x(x,t)=0,\ \ 0<t<1;\\&\tau_2(0)=0,\quad \int_0^1\tau_2(t)dt=\varphi_3(0)\end{aligned}\right\} \quad (18)$$

munosabatlarga ega bo‘lamiz.

(11) belgilashni hamda (17) va (4) tengliklarni e’tiborga olsak, (18) tengliklardan $\tau_2(t)$ noma’lum funksiyaga nisbatan

$$\begin{aligned}&\tau_2''(t)-a_1(t)D_{0t}^{1/2}\tau_2(t)+\\&+a_2(t)D_{0t}^{\alpha}[b_2(t)\tau_2(t)]+a_3(t)D_{t1}^{\beta}[b_3(t)\tau_2(t)]= \quad (19)\\&=-b_1(t)-a_1(t)D_{0t}^{1/2}\Phi_1(t),\quad 0<t<1\end{aligned}$$

ko‘rinishdagi integro-differensial tenglama va ushbu

$$\tau_2(0)=0,\qquad \int_0^1\tau_2(t)\,dt=\varphi_3(0) \quad (20)$$

shartlar kelib chiqadi.

{(19),(20)} masala yechimining mavjudligi va yagonaligini isbotlaymiz. Avval bir jinsli masalani qaraymiz:

$$\begin{aligned}&\tau_2''(t)-a_1(t)D_{0t}^{1/2}\tau_2(t)+\\&+a_2(t)D_{0t}^{\alpha}[b_2(t)\tau_2(t)]+a_3(t)D_{t1}^{\beta}[b_3(t)\tau_2(t)]=0,\quad 0<t<1;\end{aligned} \quad (21)$$

$$\tau_2(0)=0,\qquad \int_0^1\tau_2(t)dt=0 \quad (22)$$

Lemma. Agar $[0,1]$ oraliqda $a_1(t)>0,\ a_2(t)\le 0,\ a_3(t)\le 0$ va $b_2(t)>0,\ b_3(t)>0$ tengsizliklar bajarilib, $b_2(t)-$kamaymaydigan funksiya, $b_3(t)-$ o‘smaydigan funksiya bo‘lsa, {(21), (22)} masala faqat trivial yechimga ega bo‘ladi.

Isbot. (22) shartlarning ikkinchisidagi integralga o‘rta qiymat haqidagi teoremani tatbiq qilsak, $(0,1]$ oraliqda shunday t_1 nuqta mavjudki, $\tau_2(t_1)=0$ tenglik o‘rinli bo‘ladi.

Buni e’tiborga olib,

$$\left.\begin{array}{c} \tau_2''(t) - a_1(t)D_{t_1t}^{1/2}\tau_2(t) + \\ +a_2(t)D_{t_1t}^{\alpha}[b_2(t)\tau_2(t)] + a_3(t)D_{t1}^{\beta}[b_3(t)\tau_2(t)] = 0, \quad t_1 < t < 1; \\ \tau_2(t_1) = 0, \qquad \int\limits_{t_1}^{1}\tau_2(t)dt = 0 \end{array}\right\} \quad (23)$$

masalani qaraymiz.

Faraz qilaylik (23) masala $\tau_2(t) \neq 0,\ 0 \leq t \leq t_1$ yechimga ega bo'lsin. U holda $\sup\limits_{[0,1]} |\tau_2(t)| = |\tau_2(\xi)| \neq 0,\ \xi = const \in [0, t_1]$ bo'ladi. $\tau_2(0) = \tau_2(t_1) = 0$ shartlarga asosan $\xi \neq 0,\ \ \xi \neq t_1$. Demak, $\xi \in (0, t_1)$. Unda $\tau_2(t)$ funksiya $t = \xi$ nuqtada musbat maksimumga yoki manfiy minumumga erishadi. Buni hamda lemma shartlarini hamda butun hosilalarning xossalarini va kasr tartibli differensial operatorlar uchun ekstremum prinsipini e'tiborga olsak, $\tau_2(\xi) -$ musbat maksimum (manfiy minimum) bo'lganda quyidagi tengsizliklar o'rinli bo'ladi:

$$\tau_2(\xi) \leq 0 (\geq 0),\ a_1(\xi)D_{0t}^{1/2}\tau_2(t)|_{t=\xi} > 0 (< 0),$$

$$a_2(t)\,D_{0t}^{\alpha}[b_2(t)\tau_2(t)]|_{t=\xi} \leq 0 (\geq 0),$$

$$a_3(t)\,D_{0t_1}^{\beta}[b_3(t)\tau_2(t)]|_{t=\xi} \leq 0 (\geq 0).$$

Bularga ko'ra

$$\tau_2''(\xi) - \{a_1(t)D_{0t}^{1/2}\tau_2(t) -$$
$$-a_2(t)D_{0t}^{\alpha}[b_2(t)\tau_2(t)] - a_3(t)D_{tt_1}^{\beta}[b_3(t)\tau_2(t)]\}|_{t=\xi} < 0 (> 0)$$

bo'ladi. Bu tengsizlik (23) munosabatlarning birinchisiga ziddir. Biz duch kelgan bu qarama-qarshilik $\tau_2(t) \neq 0,\ 0 \leq t \leq t_1$ deb qilgan farazimiz noto'g'ri ekanligini ko'rsatadi. Demak, (23) masala faqat $\tau_2(t) \equiv 0,\ t \in [0, t_1]$ yechimga ega. Agar $t_1 = 1$ bo'lsa, lemma isbot bo'ladi. Aks holda, (21) (22) va $\tau_2(t) \equiv 0,\ \ t_1 \in [0, t_1]$ tengliklarga asoslanib,

$$\left.\begin{array}{c} \tau_2''(t) - a_1(t)D_{t_1t}^{1/2}\tau_2(t) + \\ +a_2(t)D_{t_1t}^{\alpha}[b_2(t)\tau_2(t)] + a_3(t)D_{t1}^{\beta}[b_3(t)\tau_2(t)] = 0, \quad t_1 < t < 1; \\ \tau_2(t_1) = 0, \qquad \int\limits_{t_1}^{1}\tau_2(t)dt = 0 \end{array}\right\} \quad (24)$$

masalani qaraymiz.

Yuqoridagi usulni qo'llab, (24) masala uchun ham $(t_1, 1]$ oraliqda shunday t_2 nuqta mavjudki, $\tau_2(t) \equiv 0$, $t \in [t_1, t_2]$ bo'lishini topamiz.

Agar $t_2 = 1$ bo'lsa, lemma hal bo'ladi. Aks holda, (24) va $\tau_2(t) \equiv 0,\ t \in [t_1, t_2]$ tengliklarga asoslanib,

$$\left.\begin{array}{c}\tau_2''(t)-a_1(t)D_{t_2t}^{1/2}\tau_2(t)+\\ +a_2(t)D_{t_2t}^{\alpha}[b_2(t)\tau_2(t)]+a_3(t)D_{t1}^{\beta}[b_3(t)\tau_2(t)]=0,\quad t_1<t<1;\\ \tau_2(t_2)=0,\qquad \int_{t_2}^{1}\tau_2(t)dt=0\end{array}\right\}$$

masalani qaraymiz.

Bu masalaga ham yuqoridagi usulni qo‘llab, $\exists t_3\in(t_2,1],\ \ \tau_2(t)\equiv 0,\ \ t\in[t_2,t_3]$ tenglik o‘rinli bo‘lishini topamiz.

Agar $t_3=1$ bo‘lsa, lemma hal bo‘ladi. Aks holda, yuqoridagi jarayonni davom ettirib, sanoqli qadamdan so‘ng, yoki lemmaning tasdigiga kelamiz, yoki shunday $[0,t_1],\ [t_1,t_2],...[t_{n-1},t_n],...$ oraliqlar ketma-ketligiga ega bo‘lamizki (bu yerda $t_0=0$), bunda $\tau_2(t)\equiv 0,\ t\in[t_{n-1},t_n],\ n\in N$ va $\lim\limits_{n\to+\infty}t_n=1$ tengliklar o‘rinli bo‘ladi. Bu tengliklardan $\tau_2(t)+\int_0^1\tau_2(\eta)K_5(t,\eta)\,d\eta=0,\quad 0\le t\le 1$ bo‘lganligi uchun $\tau_2(t)\equiv 0,\ \ t\in[0,1]$ tenglik kelib chiqadi.

Lemma isbotlandi.

Lemmadan foydalanib quyidagi teoremaning o‘rinli ekanligini ko‘rsatish qiyin emas.

1-teorema. Agar lemmaning shartlari bajarilsa, {(19),(20)} masala bittadan ortiq yechimga ega bo‘lmaydi.

Endi {(19),(20)} masala yechimining mavjudligini isbotlaymiz. Shu maqsadda (19) tenglamada t ni z bilan almashtirib, so‘ngra $[0,t]$ oraliqda z bo‘yicha ikki marta ketma-ket integrallaymiz. Natijada $\tau_2(t)=0$ ekanligini hisobga olib va $\tau_2'(t)=C$ belgilashni kiritib,

$$\begin{gathered}\tau_2(t)-Ct-\int_0^t(t-s)\{a_1(s)D_{0s}^{1/2}\tau_2(s)-a_2(s)D_{0s}^{\alpha}[b_2(s)\tau_2(s)]-\\ -a_3(s)D_{s1}^{\beta}[b_3(s)\tau_2(s)]\}ds+\int_0^t(t-s)[b_1(s)+a_1(s)D_{0s}^{1/2}\Phi_1(s)]ds=0\end{gathered}\tag{25}$$

tenglikka ega bo‘lamiz.

C noma’lum sonni topish maqsadida, (20) ning ikkinchi shartida foydalanamiz, ya’ni (25) tenglikni t bo‘yicha $[0,1]$ oraliqda integrallab, uni $\varphi_3(0)$ ga tenglaymiz. Natijada hosil bo‘lgan tenglikdan C noma’lum son bir qiymatli topiladi:

$$\begin{gathered}C=\{2\varphi_2(0)-\int_0^1(1-s)[a_1(s)D_{0s}^{1/2}\tau_2(s)-a_2(s)D_{0s}^{\alpha}[b_2(s)\tau_2(s)]]-\\ -a_3(s)D_{s1}^{\beta}[b_3(s)\tau_2(s)]+\int_0^1(1-s)^2[b_1(s)+a_1(s)D_{0s}^{1/2}\Phi_1(s)]ds\}.\end{gathered}$$

C sonning bu ifodasi (25) tenglikka qo‘yib, kasr tartibli hosilalar yoyilmasidan, bo‘laklab integrallash qoidasidan va takroriy integrallarda integrallash tartibini o‘zgartirish qoidasidan foydalanib, , {(19), (20)} masalaga ekvivalent bo‘lgan quyidagi ikkinchi tur Fredgolm integral tenglamasini hosil qilamiz:

$$\tau_2(t)+\int_0^1 \tau_2(\eta)K_5(t,\eta)d\eta=\Phi_5(t),\quad 0\le t\le 1, \quad (26)$$

bu yerda

$$\Phi_5(t)=\{2\varphi_3(0)+t\int_0^1(1-s)^2[b_1(s)+a_1(s)D_{0s}^{1/2}\Phi(s)]\,ds\}+$$

$$+\int_0^t(t-s)[b_1(s)+a_1(s)D_{0s}^{1/2}\Phi_1(s)]ds,$$

$$K_5(t,\eta)=\begin{cases}K_6(t,\eta), & t>\eta;\\ K_7(t,\eta), & t<\eta\end{cases}$$

$$K_6(t,\eta)=\frac{t}{\sqrt{\pi}}\int_\eta^1\frac{[2(1-s)a_1(s)-(1-s)^2a_1'(s)]}{(s-\eta)^{1/2}\,ds}-$$

$$-\frac{tb_2(\eta)}{\Gamma(1-\alpha)}\int_\eta^1(s-\eta)^{-\alpha}[2(1-s)a_2(s)-(1-s)^2a_2'(s)]ds+$$

$$+\frac{tb_3(\eta)}{\Gamma(1-\beta)}\int_0^\eta(\eta-s)^{-\beta}[2(1-s)a_3(s)-(1-s)^2a_3'(s)]ds-$$

$$-\frac{1}{\sqrt{\pi}}\int_\eta^t(s-\eta)^{-1/2}[a_1(s)-(t-s)a_1'(s)]ds+$$

$$+\frac{b_2(\eta)}{\Gamma(1-a)}\int_\eta^t(s-\eta)^{-\alpha}[a_2(s)-(t-s)a_2'(s)]ds+$$

$$-\frac{b_3(\eta)}{\Gamma(1-\beta)}\int_0^\eta(\eta-s)^{-\beta}[a_3(s)-(t-s)a_3'(s)]ds,$$

$$K_7(t,\eta)=\frac{t}{\sqrt{\pi}}\int_\eta^1\frac{[2(1-s)a_1(s)-(1-s)^2a_1'(s)]}{(s-\eta)^{1/2}\,ds}-$$

$$-\frac{tb_2(\eta)}{\Gamma(1-\alpha)}\int_\eta^t(s-\eta)^{-\alpha}[2(1-s)a_2(s)-(1-s)^2a_2'(s)]ds+$$

$$+\frac{tb_3(\eta)}{\Gamma(1-\beta)}\int_0^\eta(\eta-s)^{-\beta}[2(1-s)a_3(s)-(1-s)^2a_3'(s)]ds-$$

$$-\frac{b_3(\eta)}{\Gamma(1-\beta)}\int_0^t(\eta-s)^{-\beta}[a_3(s)-(t-s)a_3'(s)]ds.$$

Berilgan funksiyalarga qo‘yilgan shartlardan foydalanib ko‘rsatish mumkinki, $\Phi_5(t)\in C[0,1]$; $K_5(t,\eta)$ funksiya esa $\{(t,\eta):0\le t,\eta\le 1\}$ kvadratning $t\ne\eta$ bo‘lgan nuqtalarida uzluksiz bo‘lib, $t=\eta$ diogonalida birinchi tur sakrashga ega.

$$\tau_2(t)+\int_0^1\tau_2(\eta)K_5(t,\eta)\,d\eta=0,\quad 0\le t\le 1 \quad (27)$$

bir jinsli integral tenglama {(21), (22)} bir jinsli masalaga mos keladi. Bu bir jinsli masala faqat trivial yechimga ega bo‘lganligi uchun, (27) bir jinsli integral tenglama ham faqat trivial yechimga ega. U holda Fredgolm alternativasiga asosan (26) bir jinslimas integral tenglamaning yechimi mavjud va yagonadir.

$\tau_2(t)$ funksiya (26) integral tenglamadan topilgandan so‘ng, $\tau_3(t)$ funksiya

$$\tau_3(t)=-\frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t(t-\eta)^{-1/2}\left[\tau_2(\eta)-\Phi_1(\eta)\right]d\eta,\ t\in(0,1) \quad (28)$$

(28) tenglik yordamida aniqlanadi. Shundan so‘ng, masalaning yechimi D_0 sohada (16) formula orqali topiladi, D_2 sohada esa $u_{tt}+u_x=0$ tenglamaning (3), (6), $u(0,t)=\tau_2(t)$, $0\le t\le 1$ shartlarni qanoatlantiruvchi yechimi sifatida aniqlanadi. Shuning uchun oxirigi masalani J_1 bilan belgilaymiz va o‘rganamiz.

J_1 masala yechimini

$$u(x,t)=\int_0^1\tau_2(\eta)G_1(x,t;0,\eta)\,d\eta+\int_x^0\varphi_2(\xi)G_{1\eta}(x,t;\xi,0)\,d\xi-\int_x^0\varphi(\xi)G_{1\eta}(x,t;\xi,1)\,d\xi \quad (29)$$

(29) ko‘rinishda izlaymiz, bu yerda $\varphi(x)=u(x,1)-$ noma’lum funksiya. (29) funksiyani integrallab,

$$\int_0^1 u(x,t)\,dt=-\int_x^0\varphi(\xi)\left\{\int_0^1 G_{1\eta}(x,t;\xi,1)\,dt\right\}d\xi+\Phi_4(x). \quad (30)$$

$$\int_0^1 G_{1\eta}(x,t;\xi,1)\,dt=-\frac{1}{\sqrt{\pi(\xi-x)}}-K_4(x,\xi), \quad (31)$$

bu yerda

$$K_4(x,\xi)=\frac{2}{\sqrt{\pi(\xi-x)}}\sum_{n=1}^{+\infty}\left\{\exp\left[-\frac{n^2}{\xi-x}\right]-\exp\left[-\frac{(2n-1)^2}{4(\xi-x)}\right]\right\}$$

(30) va (31) tengliklarga asosan quyidagi tenglikka ega bo‘lamiz:

$$\int_0^1 u(x,t)\,dt=\int_x^0\left[\frac{1}{\sqrt{\pi(\xi-x)}}+K_4(x,\xi)\right]\varphi(\xi)\,d\xi+\Phi_4(x).$$

Buni (5) shartga qo‘yib,

$$\int_x^0\frac{\varphi(\xi)\,d\xi}{\sqrt{\pi(\xi-x)}}=\varphi_2(x)-\Phi_4(x)-\int_x^0\varphi(\xi)K_4(x,\xi)\,d\xi \quad (32)$$

ko‘rinishdagi integral tenglamaga kelamiz, bu yerdagi $K_4(x,\xi)$ quyidagi (32) tenglik bilan aniqlanib, bu funksiya va uning ixtiyoriy tartibli hosilalari $x<\xi$ bo‘lganda uzluksiz, $x\to\xi$ da nolga intiladi.

$$K_4(x,\xi)=\frac{2}{\sqrt{\pi(\xi-x)}}\sum_{n=1}^{+\infty}\left\{\exp\left[-\frac{n^2}{\xi-x}\right]-\exp\left[-\frac{(2n-1)^2}{4(\xi-x)}\right]\right\} \quad (33)$$

(32) tenglamaning o‘ng tomonini vaqtincha ma’lum funksiya desak, u noma’lum funksiyaga nisbatan Abel integral tenglamasi bo‘ladi. U holda (32) tenglamadan, Abel integral tenglamasi yechimining formulasiga ko‘ra,

$$\varphi(x)=\frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_x^0\frac{dz}{\sqrt{z-x}}\left\{\varphi_3(z)-\Phi_4(z)-\int_z^0\varphi(\xi)K_4(x,\xi)\,d\xi\right\} \quad (34)$$

tenglik kelib chiqadi. Bu yerda avval integrallash tartibini o‘zgartirib, so‘ngra hosila olish amalini bajarsak,

$$\varphi(x)-\int_x^0\varphi(\xi)K_8(x,\xi)\,d\xi=\Phi_6(x),\qquad x\in(-1,0) \quad (35)$$

ko‘rinishdagi integral tenglama hosil bo‘ladi, bunda

$$K_8(x,\xi)=\frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_x^{\xi}K_4(z,\xi)(z-x)^{-1/2}dz,$$

$$\Phi_6(x)=-\frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_x^0[\varphi_2(z)-\Phi_4(z)](z-x)^{-1/2}dz$$

(35) integral tenglamaning yadrosini va o‘ng tomonini o‘rganamiz. Avval $K_8(x,\xi)$ yadroni qaraymiz. Bu funksiyaning tarkibidagi integralni bo‘laklab va $K'_{4\xi}(x,\xi)\in C(-1\le x<\xi\le 0)$, $\lim\limits_{x\to\xi}K_4(x,\xi)=0$, $\lim\limits_{x\to\xi}K'_{4\xi}(x,\xi)=0$ munosabatlarni hisobga olib, ishonch hosil qilish mumkinki, $K_8(x,\xi)\in C(-1\le x<\xi\le 0)$ va $K_8(x,\xi)=(\xi-x)^{-1/2}O(1)$.

Endi $\Phi_6(x)$ funksiyani tekshiraylik. Buning uchun $\Phi_4(x)$ funksiyani $\Phi_6(x)$ga qo‘yib, uni quyidagicha yozib olamiz:

$$\Phi_6(x)=\Phi_{61}(x)-\Phi_{62}(x)-\Phi_{63}(x), \quad (36)$$

bu yerda

$$\Phi_{61}(x)=-\frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_x^0\varphi_3(z)(z-x)^{-1/2}dz,$$

$$\Phi_{62}(x)=-\frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_x^0(z-x)^{-1/2}dz\int_z^0\varphi_2(\xi)d\xi\int_0^1G_{1\eta}(z,t;\xi,0)\,dt.$$

$$\Phi_{63}(x)=-\frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_x^0(z-x)^{-1/2}dz\int_0^1\tau_2(\eta)d\eta\int_0^1G_1(z,t;0,\eta)\,dt.$$

$\Phi_{6j}(x),\ j=\overline{1,3}$ funksiyalarni alohida-alohida o‘rganamiz.

Bo‘laklab integrallash formulasini qo‘llasak, $\Phi_{61}(x)$ funksiya quyidagicha yoziladi:

$$\Phi_{61}(x)=-\frac{\varphi_3(0)}{\sqrt{-\pi x}}-\frac{1}{\sqrt{\pi}}\int_x^0\varphi_3'(z)(z-x)^{-1/2}dz. \quad (37)$$

$$\Phi_{62}(x)=-\frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_x^0\varphi_3(\xi)d\xi\int_x^\xi(z-x)^{-1/2}dz\int_0^1 G_{1\eta}(z,t;\xi,0)\,dt.$$

$$\int_0^T G_{1\eta}(x,t;\xi,\eta)\,dt=\int_0^T\frac{1}{2\sqrt{\pi(\xi-x)}}\frac{\partial}{\partial\eta}\sum_{n=-\infty}^{\infty}\left\{\exp\left[\frac{(t-\eta-2nT)^2}{-4(\xi-x)}\right]-\exp\left[\frac{(t+\eta-2nT)^2}{-4(\xi-x)}\right]\right\}dt=$$

$$=\frac{-1}{2\sqrt{\pi(\xi-x)}}\int_0^T\frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty}\left\{\exp\left[\frac{(t-\eta-2nT)^2}{-4(\xi-x)}\right]-\exp\left[\frac{(t+\eta-2nT)^2}{-4(\xi-x)}\right]\right\}dt=$$

$$=\frac{-1}{2\sqrt{\pi(\xi-x)}}\sum_{n=-\infty}^{\infty}\left\{\exp\left[\frac{(t-\eta-2nT)^2}{-4(\xi-x)}\right]-\exp\left[\frac{(t+\eta-2nT)^2}{-4(\xi-x)}\right]\right\}\Bigg|_{t=0}^{t=T}=\frac{1}{2\sqrt{\pi(\xi-x)}}\times$$

$$\times\sum_{n=-\infty}^{\infty}\left\{\exp\left[\frac{(2nT+\eta)^2}{-4(\xi-x)}\right]+\exp\left[\frac{(2nT-\eta)^2}{-4(\xi-x)}\right]-\exp\left[\frac{(t-\eta-2nT)^2}{-4(\xi-x)}\right]-\exp\left[\frac{(t+\eta-2nT)^2}{-4(\xi-x)}\right]\right\} \quad (38)$$

Yuqoridagi (38) tenglikdan osongina kelib chiquvchi

$$\int_0^1 G_{1\eta}(x,t;0,\eta)\,dt=-\frac{1}{\sqrt{\pi(\xi-z)}}-K_4(z,\xi),$$

tenglikni e’tiborga olsak, $\Phi_{62}(x)$ funksiya

$$\Phi_{62}(x)=-\frac{1}{\pi}\frac{d}{dx}\int_x^0\varphi_2(\xi)\,d\xi\int_x^\xi(z-x)^{-1/2}[(\xi-z)^{-1/2}dz+\sqrt{\pi}K_4(z,\xi)]dz$$

ko‘rinishda yoziladi. Ichki integralda $z=x+(\xi-x)s$ formula bilan almashtirish bajaramiz:

$$\Phi_{62}(x)=-\frac{1}{\pi}\frac{d}{dx}\int_x^0\varphi_2(\xi)d\xi\int_0^1(1-s)^{-1/2}s^{-1/2}ds-$$

$$-\frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_x^0\varphi_3(\xi)(\xi-x)^{1/2}d\xi\int_0^1 s^{-1/2}d\xi\int_0^1 s^{-1/2}K_4[x+(\xi-x)s,\xi]ds=$$

$$=\varphi_2(x)+\frac{1}{2\sqrt{\pi}}\int_x^0\varphi_2(\xi)(\xi-x)^{-1/2}d\xi\int_0^1 s^{-1/2}K_4[x+(\xi-x)s,\xi]ds-$$

$$-\frac{1}{\sqrt{\pi}}\int_x^0\varphi_2(\xi)(\xi-x)^{1/2}d\xi\int_0^1 s^{-1/2}\frac{\partial}{\partial x}K_4[x+(\xi-x)s,\xi]ds.$$

$K_4(x,\xi)$ funksiyaning xossalarini va $\varphi_3(x)\in C[-1,0]$ ekanligini inobatga olsak, oxirgi tenglikdan $\Phi_{62}(x)$ funksiyaning $[-1,0]$ segmentda uzluksiziligi kelib chiqadi.

Endi $\Phi_{63}(x)$ funksiyani qaraymiz. Bu funksiya tarkibidagi $[x,0]$ oraliq bo'yicha integralga bo'laklab integrallash formulasini qo'llab, so'ngra t bo'yicha hosila olish amalini bajarib,

$$\Phi_{63}(x)=\frac{1}{\sqrt{\pi}}\left\{\frac{1}{\sqrt{-x}}\lim_{z\to 0}\int_0^1 dt\int_0^1 \tau_2(\eta)G_1(z,t;0,\eta)\,d\eta+\right.$$

$$\left.+\int_x^0 (z-x)^{-1/2}dz\int_0^1 dt\int_0^1 \tau_2(\eta)\frac{\partial}{\partial z}G_1(z,t;0,\eta)\,d\eta\right\} \quad (39)$$

tenglikni hosil qilamiz.

Ma'lumki, quyidagi tengliklar o'rinli:

$$\lim_{z\to 0}\int_0^1 \tau_2(\eta)G_1(z,t;0,\eta)\,d\eta=\tau_2(t),$$

$$\frac{\partial}{\partial z}G_1(z,t;0,\eta)=-\frac{\partial^2}{\partial t^2}G_1(z,t;0,\eta).$$

Bularni hisobga olsak, (38) tenglik

$$\Phi_{63}(x)=\frac{1}{\sqrt{-\pi x}}\int_0^1 \tau_2(t)\,dt+\frac{1}{\sqrt{\pi}}\int_x^0 I(z)(z-x)^{-1/2}dz \quad (40)$$

ko'rinishni oladi, bu yerda

$$I(z)=-\int_0^1 \tau_2(\eta)d\eta\int_0^1 \frac{\partial^2}{\partial t^2}G_1(z,t;0,\eta)\,dt.$$

Ushbu funksiyani kiritaylik:

$$N(z,t;\eta)=\frac{1}{2\sqrt{-\pi x}}\sum_{n=-\infty}^{\infty}\left\{\exp\left[-\frac{(t-\eta-2n)^2}{(-4z)}\right]+\exp\left[-\frac{(t+\eta-2n)^2}{(-4z)}\right]\right\}$$

U holda quyidagi tengliklar o'rinli bo'ladi:

$$\frac{\partial}{\partial t}G_1(z,t;0,\eta)=-\frac{\partial}{\partial \eta}N(z,t;\eta),$$

$$N(z,1;1)=N(z,0;0)=\frac{1}{\sqrt{-\pi z}}\left[1+2\sum_{n=1}^{+\infty}\exp\left(\frac{n^2}{z}\right)\right],$$

$$N(z,1;0)=N(z,0;1)=\frac{2}{\sqrt{-\pi z}}\sum_{n=1}^{+\infty}\exp\left[-\frac{(2n-1)^2}{(-4z)}\right].$$

Bu tengliklardan foydalanib, $I(z)$ funksiyani

$$I(z)=-\int_0^1 \tau_2(\eta)\frac{\partial}{\partial t}G_1(z,t;0,\eta)\Big|_{t=0}^{t=1}d\eta=\int_0^1 \tau_2(\eta)\frac{\partial}{\partial \eta}[N(z,1;\eta)-N(z,0;\eta)]d\eta=$$

$$=\int_0^1 \tau_2{}'(\eta)[N(z,0;\eta)-N(z,1;\eta)]d\eta+[\tau_2(1)+\tau_2(0)][N(z,0;0)-N(z,0;1)]$$

ko‘rinishida yozish mumkin.

Kiritilgan $N(z,t;\eta)$ funksiya tuzilishidan kelib chiqadiki,

$$N(z,1;\eta)=\frac{1}{\sqrt{-\pi z}}\exp\left[\frac{(1-\eta)^2}{4z}\right]+K_9(z,\eta),$$

$$N(z,0;\eta)=\frac{1}{\sqrt{-\pi z}}\exp\left[\frac{\eta^2}{4z}\right]+K_{10}(z,\eta),$$

bu yerda $K_9(z,\eta)$ va $K_{10}(z,\eta)-\{(z,\eta):-1\le z<0,\ 0\le\eta\le1\}$ sohada ixtiyoriy tartibli uzluksiz va chegaralangan hosilalarga ega bo‘lgan funksiyalar.

Buni hisobga olsak, $I(z)$ funksiya quyidagicha yoziladi:

$$I(z)=\int_0^1 \tau_2{}'(\eta)\left[K_{10}(z,\eta)-K_9(z,\eta)\right]d\eta-$$

$$-\frac{1}{\sqrt{-\pi z}}\int_0^1 \tau_2{}'(\eta)\exp\left[\frac{(1-\eta)^2}{4z}\right]d\eta+\frac{1}{\sqrt{-\pi z}}\int_0^1 \tau_2{}'(\eta)\exp(\frac{\eta^2}{4z})d\eta+$$

$$+\left[\tau_2(1)+\tau_2(0)\right]\left[N(z,0;0)-N(z,0;1)\right].$$

Bu yerdagi ikkinchi va uchinchi integrallar o‘zgaruvchilarini mos ravishda $\eta=1-2s\sqrt{-z}$ va $\eta=2s\sqrt{-z}$ tengliklar yordamida almashtiramiz:

$$I(z)=\int_0^1 \tau_2{}'(\eta)[K_{10}(z,\eta)-K_9(z,\eta)]d\eta-$$

$$-\frac{2}{\sqrt{\pi}}\int_0^{1/2\sqrt{-z}}\left[\tau_2{}'(1-2\sqrt{-z}s)-\tau_2{}'(2\sqrt{-z}s)\right]e^{-s^2}ds+$$

$$+[\tau_2(1)+\tau_2(0)][N(z,0;0)-N(z,0;1)].$$

$N(z,0;0)$ va $N(z,0;1)$ funksiyalarning tuzilishiga asosan bu tenglikdan kelib chiqadiki, agar $\tau_2(t)\in C(0,1)\cap L(0,1)$ bo‘lsa,

$$I(z)\in C(-1,0],\quad I(z)=|z|^{-\delta}O(1)\quad (41)$$

munosabatlar o‘rinli bo‘ladi, bu yerda $\delta\in[0,1/2]$.

(41) munosabatlarga ko‘ra, (40) tenglikning ikkinchi qo‘shiluvchisi $C[-1,0]$ sinfga tegishli ekanligi kelib chiqadi.

(3), (6), (40) tengliklarni va $\int_0^1 \tau_2(t)\,dt = \varphi_3(0)$ shartni inobatga olsak, yuqorida isbotlanganlardan quyidagi teorema kelib chiqadi:

2-teorema. Agar $\tau_2(t) \in C[0,1] \cap C^1(0,1)$, $\tau_2{}'(t) \in L(0,1)$, $\varphi_2(x) \in C[-1,0]$, $\varphi_3(x) \in C[-1,0] \cap C^1(0,1)$, $\varphi_2{}'(x) \in L_2(-1,0)$ shartlar bajarilsa, $\Phi_6(x) \in C[-1,0]$ bo‘ladi.

Demak, (35) – sust maxsuslikka ega bo‘lgan yadroli Volterra integral tenglamasi, bo‘lib, uning o‘ng tomoni $C[-1,0]$ sinfga tegishli. Shuning uchun u uzluksiz funksiyalar sinfida yagona yechimga ega. Undan topilgan $\varphi(x)$ funksiyani (29) formulaga qo‘yib, J_1 masalaning yechimiga ega bo‘lamiz.

Shu bilan masala to‘la hal etildi.

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