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Yadrosida Bessel Funksiyasi Qatnashgan Riman-Liuvill Operatorini O'Z Ichiga Oluvchi Bir Oddiy Differensial Tenglama Uchun Koshi Masalasi

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Annotatsiya: Mazkur yadrosida Bessel funksiyasi qatnashgan Riman-Liuvil operatorni oʻz ichiga oluvchi bir oddiy differensial tenglama uchun Koshi masalasi yechimining mavjudligi va yagonalini isbotlangan.

Kalit soʻzlar: Riman-Liuvill integro-differensial operatori, Koshi masalasi, ketma-ket yaqinlashishlar usuli.

Kirish. Soʻnggi vaqtlarda tadqiqotchilar tomonidan kasr tartibli integral va differensial operatorlar va ular ishtirok etgan tenglamalarni oʻrganishga boʻlgan qiziqish ortdi. Buni bir tomondan matematika fanining ichki ehtiyoji sifatida izohlansa, ikkinchi tomondan fan va texnikaning turli muammolarini matematik modellashtirishda shunday operatorlar ishtirok etgan tenglamalar hosil boʻlishi bilan izohlash mumkin $[1],[2],[3]$. Bu yoʻnalishdagi tadqiqotlar turli yoʻnalishlarda rivojlanib bormoqda. Dastlabki tadqiqotlarda asosan Riman-Liuvill va Kaputo ma'nosidagi kasr tartibli integro-differensial operatorlar qaralgan boʻlsa, $[4]-[8]$, Soʻnggi vaqtlarda ularning turli umumnlashmalarini oʻrganishga boʻlgan qiziqish ortdi $[9],[10],[11]$. Ushbu maqolada biz yadrosida Bessel funksiyasi qatnashgan umumlashgan Riman-Liuvill operatorini oʻz ichiga oluvchi oddiy differensial tenglama uchun Koshi masalasini bayon qilib, uning yechimi formulasini oshkor koʻrinishda ifodalaymiz.

2. Masalaning qoʻyilishi. Ushbu tenglamani qaraylik:

$$D_{0x}^{\alpha,\gamma} y(x) + \lambda y(x) = f(x), \;\; x \in (0,T), \qquad (1)$$

bu yerda $y(x)$-noma'lum funksiya, $f(x)$-berilgan funksiya, α,γ,λ,T lar esa berilgan haqiqiy sonlar boʻlib, $3<\alpha<4$, $T>0$; $D_{ox}^{\alpha,\gamma}$ esa differensial operator boʻlib, ushbu munosabat bilan aniqlanadi [11]:

$$D_{0x}^{\alpha,\gamma} y(x) = \left(\frac{d^2}{dx^2}+\gamma^2\right)^2 I_{0x}^{4-\alpha,\gamma} y(x), \qquad (2)$$

$$I_{0x}^{\beta,\gamma} y(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} \overline{J}_{(\beta-1)/2}[\gamma(x-t)] y(t)dt, \qquad (3)$$

$\overline{J}_\nu(z)$-Bessel-Kliford funksiyasi boʻlib, quyidagi tenglik bilan aniqlanadi:

$$\overline{J}_\nu(z) = \Gamma(\nu+1)\left(z/2\right)^{-\nu} J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k!(\nu+1)_k}, \qquad (4)$$

$(z)_k$-Poxgammer belgisi, $\Gamma(x)$-Eylerning gamma funksiyasi [12], $J_\nu(x)$-birinchi tur Bessel funksiyasi [13].

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$D_{0x}^{\alpha,\gamma}y(x)$ va $I_{0x}^{\beta,\gamma}y(x)$ operatorlar [11] ishda kiritilgan va ularning xossalari o'rganilgan bo'lib, ular mos holda Riman-Liuvill ma'nosidagi kasr tartibli differensial va integral operatorlarining umumlashmasidir.

1. Tenglama uchun quyidagi masalani o'rganamiz:
   Koshi masalasi. (1) tenglamani va quyidagi

$$\lim_{x\to 0} I_{0x}^{4-\alpha,\gamma}y(x)=A_1, \qquad \lim_{x\to 0}\left(\frac{d^2}{dx^2}+\gamma^2\right)I_{0x}^{4-\alpha,\gamma}y(x)=A_3,$$
$$\lim_{x\to 0}\frac{d}{dx}I_{0x}^{4-\alpha,\gamma}y(x)=A_2, \qquad \lim_{x\to 0}\frac{d}{dx}\left(\frac{d^2}{dx^2}+\gamma^2\right)I_{0x}^{4-\alpha,\gamma}y(x)=A_4 \tag{5}$$

boshlang'ich shartlarni qanoatlantiruvchi $y(x)$ funksiya topilsin, bu yerda A_1, A_2, A_3, A_4-berilgan haqiqiy sonlar[11].
Quyidagi teorema o'rinli:

1-teorema. Agar $f(x)=x^{-p}f_1(x),\ f_1(x)\in C[0,T], 0\le p<\alpha-1,$ bo'lsa $\{(1),(5)\}$ Koshi masalasining yechimi mavjud va u quyidagi

$$y(x)=A_1x^{\alpha-4}\mathrm{E}_{\alpha,\alpha-3,(\alpha-5)/2}\left[-\lambda x^{\alpha};\gamma x\right]+A_2x^{\alpha-3}\mathrm{E}_{\alpha,\alpha-2,(\alpha-3)/2}\left[-\lambda x^{\alpha};\gamma x\right]+$$
$$+A_3x^{\alpha-2}\mathrm{E}_{\alpha,\alpha-1,(\alpha-3)/2}[-\lambda x^{\alpha};\gamma x]+A_4x^{\alpha-1}\mathrm{E}_{\alpha,\alpha,(\alpha-1)/2}[-\lambda x^{\alpha};\gamma x]+$$
$$+\int_0^x (x-z)^{\alpha-1}\mathrm{E}_{\alpha,\alpha,(\alpha-1)/2}\left[-\lambda(x-z)^{\alpha};\gamma(x-z)\right]f(z)dz \tag{6}$$

formula bilan aniqlanadi [14], bu yerda

$$\mathrm{E}_{\alpha,\beta,\theta}[x;y]=\sum_{n=0}^{+\infty}\frac{x^n}{\Gamma(\alpha n+\beta)}\bar{J}_{\alpha n/2+\theta}(y). \tag{7}$$

$\alpha>0,\beta>0$ (7) tenglik bilan aniqlangan qatorlar uchun $-\infty<x,y<\infty$ bo'lganda absolyut va tekis yaqinlashuvchi bo'ladi [15].
Bundan tashqari (7) funksiya uchun quyidagi tengliklar

$$\mathrm{E}_{\alpha,\beta,\theta}[x;0]=E_{\alpha,\beta}(x), \mathrm{E}_{\alpha,\beta,\theta}[0;y]=\frac{1}{\Gamma(\beta)}\bar{J}_{\theta}(y), \mathrm{E}_{\alpha,\beta,\theta}[0,0]=\frac{1}{\Gamma(\beta)}$$

va quyidagi hosilani hisoblash formulalari o'rinli:

$$\frac{d}{dx}\mathrm{E}_{\alpha,1,(-1/2)}\left[-\lambda x^{\alpha};\gamma x\right]=-\lambda x^{\alpha-1}\mathrm{E}_{\alpha,\alpha,(\alpha-1)/2}\left[-\lambda x^{\alpha};\gamma x\right]-\gamma^2 x\mathrm{E}_{\alpha,2,1/2}\left[-\lambda x^{\alpha};\gamma x\right], \tag{8}$$

$$\frac{d}{dx}\left\{x^{\beta-1}\mathrm{E}_{\alpha,\beta,(\beta-1)/2}\left[-\lambda x^{\alpha};\gamma x\right]\right\}=x^{\beta-2}\mathrm{E}_{\alpha,\beta-1,(\beta-3)/2}\left[-\lambda x^{\alpha};\gamma x\right],\beta\ne 1. \tag{9}$$

Endi teoremaning isbotiga o'tamiz. Shu maqsadda dastlab (1) tenglamaga $I_{0x}^{\alpha,\gamma}y(x)$ operatorni ta'sir ettiramiz, so'ngra ushbu

$$\left(I_{a+}^{\alpha,\lambda}D_{a+}^{\alpha,\lambda}f\right)(x)=f(x)-\frac{1}{\Gamma(\alpha)}\sum_{k=o}^{m}\left(\frac{d^2}{dx^2}+\lambda^2\right)^k\times$$
$$\times\left\{-(x-a)^{\alpha-1}\bar{J}_{(\alpha-1)/2}\left[\lambda(x-a)\right]\left[\frac{d}{dt}\left(\frac{d^2}{dt^2}+\lambda^2\right)^{m-k}I_{a+}^{2m+2-\alpha,\lambda}f\right]_{t=a}\right. +$$

$$+(1-\alpha)(x-a)^{\alpha-2}\bar{J}_{(\alpha-3)/2}\left[\lambda(x-a)\right]\left[\left(\frac{d^2}{dx^2}+\lambda^2\right)^{m-k}I_{a+}^{2m+2-\alpha,\lambda}f(t)\right]_{t=a}\Bigg\}$$

tenglikni e'tiborga olib [11], quyidagiga ega bo'lamiz:

$$I_{0x}^{\alpha,\gamma}D_{0x}^{\alpha,\gamma}y(x)=y(x)-\frac{x^{\alpha-1}}{\Gamma(\alpha)}\bar{J}_{(\alpha-1)/2}(\gamma x)\lim_{x\to0}\frac{d}{dx}\left(\frac{d^2}{dx^2}+\gamma^2\right)I_{0x}^{4-\alpha,\gamma}y(x)-$$

$$-\frac{x^{\alpha-2}}{\Gamma(\alpha-1)}\bar{J}_{(\alpha-3)/2}(\gamma x)\lim_{x\to0}\left(\frac{d^2}{dx^2}+\gamma^2\right)I_{0x}^{4-\alpha,\gamma}y(x)-\frac{1}{\Gamma(\alpha)}\left(\frac{d^2}{dx^2}+\gamma^2\right)x^{\alpha-1}\bar{J}_{(\alpha-1)/2}(\gamma x)\lim_{x\to0}\frac{d}{dx}I_{0x}^{4-\alpha,\gamma}y(x)-$$

$$-\frac{1}{\Gamma(\alpha-1)}\left(\frac{d^2}{dx^2}+\gamma^2\right)x^{\alpha-2}\bar{J}_{(\alpha-1)/2}(\gamma x)\lim_{x\to0}I_{0x}^{4-\alpha,\gamma}y(x).$$

(5) shartlarni e'tiborga olsak, oxirgi tenglik quyidagi ko'rinishni oladi:

$$y(x)+\lambda I_{ox}^{\alpha,\gamma}y(x)=I_{ox}^{\alpha,\gamma}f(x)+$$

$$+A_1\frac{1}{\Gamma(\alpha-1)}\left(\frac{d^2}{dx^2}+\gamma^2\right)x^{\alpha-2}\bar{J}_{(\alpha-3)/2}(\gamma x)+A_2\frac{1}{\Gamma(\alpha)}\left(\frac{d^2}{dx^2}+\gamma^2\right)x^{\alpha-1}\bar{J}_{(\alpha-1)/2}(\gamma x)+$$

$$+A_3\frac{x^{\alpha-2}}{\Gamma(\alpha-1)}\bar{J}_{(\alpha-3)/2}(\gamma x)+A_4\frac{x^{\alpha-1}}{\Gamma(\alpha)}\bar{J}_{(\alpha-1)/2}(\gamma x).\qquad(10)$$

Dastlab $\left(\frac{d^2}{dx^2}+\gamma^2\right)x^{\alpha-2}\bar{J}_{(\alpha-3)/2}(\gamma x)$ va $\left(\frac{d^2}{dx^2}+\gamma^2\right)x^{\alpha-1}\bar{J}_{(\alpha-1)/2}(\gamma x)$ ifodalarni soddalashtiraylik. $\bar{J}_{\frac{\alpha-3}{2}}(\gamma x)$ Bessel-kliford funksiyasini uning (4) ifodasi bilan almashtirsak va kerakli hosilalarni hisoblasak, quyidagi natijaga ega bo'lamiz:

$$\left(\frac{d^2}{dx^2}+\gamma^2\right)x^{\alpha-2}\bar{J}_{(\alpha-3)/2}(\gamma x)=\frac{d^2}{dx^2}\sum_{k=0}^{\infty}\frac{(-1)^k(\gamma/2)^{2k}x^{2k+\alpha-2}}{k!\left((\alpha-1)/2\right)_k}+\gamma^2\sum_{k=0}^{\infty}\frac{(-1)^k(\gamma/2)^{2k}x^{2k+\alpha-2}}{k!\left((\alpha-1)/2\right)_k}=$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k(\gamma/2)^{2k}(2k+\alpha-2)(2k+\alpha-3)x^{2k+\alpha-4}}{k!\left((\alpha-1)/2\right)_k}+\gamma^2\sum_{k=0}^{\infty}\frac{(-1)^k(\gamma/2)^{2k}x^{2k+\alpha-2}}{k!\left((\alpha-1)/2\right)_k}.(11)$$

Ma'lumki, ushbu tengliklar o'rinli:

$$\Gamma(a+n)=(a)_n\Gamma(a),\ (a)_{2n}=2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n.\qquad(12)$$

(11) ning o'ng tomonidagi yig'indining birinchi hadi uchun (12) formulani ketma-ket qo'llasak, quyidagi tenglik hosil bo'ladi:

$$\left(\frac{d^2}{dx^2}+\gamma^2\right)x^{\alpha-2}\bar{J}_{(\alpha-3)/2}(\gamma x)=$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k(\gamma/2)^{2k}(2k+\alpha-2)(2k+\alpha-3)\Gamma\left((\alpha-3)/2\right)\left((\alpha-3)/2\right)x^{2k+\alpha-4}}{k!\left((2k+\alpha-3)/2\right)\Gamma\left((\alpha-3)/2+k\right)}+\gamma^2\sum_{k=0}^{\infty}\frac{(-1)^k(\gamma/2)^{2k}x^{2k+\alpha-2}}{k!\left((\alpha-1)/2\right)_k}.$$

Bundan esa ba'zi hisoblashlardan so'ng, ushbu

$$A_1\frac{1}{\Gamma(\alpha-1)}\left(\frac{d^2}{dx^2}+\gamma^2\right)x^{\alpha-2}\bar{J}_{(\alpha-3)/2}(\gamma x)=A_1\frac{x^{\alpha-4}}{\Gamma(\alpha-3)}\bar{J}_{(\alpha-5)/2}(\gamma x)$$

natijaga ega bo'lamiz.

$\left(\dfrac{d^2}{dx^2}+\gamma^2\right)x^{\alpha-1}\overline{J}_{(\alpha-1)/2}(\gamma x)$ funksiya uchun ifodani ham yuqoridagi kabi aniqlash mumkin. Topilgan natijalarni (10) tenglikka qo‘yib, quyidagi Volterraning ikkinchi tur integral tenglamasiga ega bo‘lamiz:

$$y(x)+\lambda I_{0x}^{\alpha,\gamma}y(x)=I_{0x}^{\alpha,\gamma}f(x)+A_1\frac{x^{\alpha-4}}{\Gamma(\alpha-3)}\overline{J}_{(\alpha-5)/2}(\gamma x)+A_2\frac{x^{\alpha-3}}{\Gamma(\alpha-2)}\overline{J}_{(\alpha-3)/2}(\gamma x)+$$
$$+A_3\frac{x^{\alpha-2}}{\Gamma(\alpha-1)}\overline{J}_{(\alpha-3)/2}(\gamma x)+A_4\frac{x^{\alpha-1}}{\Gamma(\alpha)}\overline{J}_{(\alpha-1)/2}(\gamma x).\qquad(13)$$

(13) integral tenglamani yechish uchun ketma-ket yaqinlashishlar usulidan foydalanamiz [14]. Nolinchi yaqinlashishni ushbu

$$y_0(x)=I_{0x}^{\alpha,\gamma}f(x)+A_1\frac{x^{\alpha-4}}{\Gamma(\alpha-3)}\overline{J}_{(\alpha-5)/2}(\gamma x)+A_2\frac{x^{\alpha-3}}{\Gamma(\alpha-2)}\overline{J}_{(\alpha-3)/2}(\gamma x)+$$
$$+A_3\frac{x^{\alpha-2}}{\Gamma(\alpha-1)}\overline{J}_{(\alpha-3)/2}(\gamma x)+A_4\frac{x^{\alpha-1}}{\Gamma(\alpha)}\overline{J}_{(\alpha-1)/2}(\gamma x)$$

tenglik bilan, qolgan yaqinlashishlarni esa

$$y_m(x)=y_0(x)-\lambda I_{0x}^{\alpha,\gamma}y_{m-1}(x),\ m\in N$$

munosabat bilan aniqlaymiz.

$I_{ax}^{\alpha,\gamma}I_{ax}^{\beta,\gamma}\varphi(x)=I_{ax}^{\beta,\gamma}I_{ax}^{\alpha,\gamma}\varphi(x)=I_{ax}^{\alpha+\beta,\gamma}\varphi(x)$ formuladan foydalanib, $y_m(x)$ ni quyidagi ko‘rinishini yozib olamiz [14]:

$$y_m(x)=y_0(x)-\lambda I_{0x}^{\alpha,\gamma}y_0(x)+\lambda^2I_{ox}^{2\alpha,\gamma}y_0(x)-\lambda^3I_{0x}^{3\alpha,\gamma}y(x)+...+(-\lambda)^mI_{0x}^{\alpha m,\gamma}y_0(x).\ (14)$$

(14) tenglikka $y_0(x)$ ning ifodasini qo‘yib, $m\to\infty$ da limitga o‘tsak,

$$y(x)=\frac{A_1}{\Gamma(\alpha-3)}\sum_{n=0}^{\infty}(-\lambda)^nI_{0x}^{\alpha n,\gamma}\{x^{\alpha-4}\overline{J}_{(\alpha-5)/2}(\gamma x)\}+\frac{A_2}{\Gamma(\alpha-2)}\sum_{n=0}^{\infty}(-\lambda)^nI_{0x}^{\alpha n,\gamma}\{x^{\alpha-3}\overline{J}_{(\alpha-3)/2}(\gamma x)\}+$$
$$+\frac{A_3}{\Gamma(\alpha-1)}\sum_{n=0}^{\infty}(-\lambda)^nI_{0x}^{\alpha n,\gamma}\{x^{\alpha-2}\overline{J}_{(\alpha-3)/2}(\gamma x)\}+\frac{A_4}{\Gamma(\alpha)}\sum_{n=0}^{\infty}(-\lambda)^nI_{0x}^{\alpha n,\gamma}\{x^{\alpha-1}\overline{J}_{(\alpha-1)/2}(\gamma x)\}+$$
$$+\sum_{n=0}^{\infty}(-\lambda)^nI_{0x}^{\alpha n+\alpha,\gamma}f(x)\ (15)$$

tenglik kelib chiqadi.

(15) tenglikni soddalashtirish maqsadida

$I_{0x}^{\alpha n,\gamma}\{x^{\alpha-4}\overline{J}_{(\alpha-5)/2}(\gamma x)\},I_{0x}^{\alpha n,\gamma}\{x^{\alpha-3}\overline{J}_{(\alpha-3)/2}(\gamma x)\},I_{0x}^{\alpha n,\gamma}\{x^{\alpha-2}\overline{J}_{(\alpha-3)/2}(\gamma x)\}$,

$I_{0x}^{\alpha n,\gamma}\{x^{\alpha-1}\overline{J}_{(\alpha-1)/2}(\gamma x)\}$ va $I_{0x}^{\alpha n+\alpha,\gamma}f(x)$ integrallarni qaraymiz.

$I_{0x}^{\alpha n,\gamma}\{x^{\alpha-4}\overline{J}_{(\alpha-5)/2}(\gamma x)\}$ uchun (3) ga ko‘ra, ushbu tenglik o‘rinli:

$$I_{0x}^{\alpha n,\gamma}\{x^{\alpha-4}\overline{J}_{(\alpha-5)/2}(\gamma x)\}=\frac{1}{\Gamma(\alpha n)}\int_0^x z^{\alpha-4}(x-z)^{\alpha n-1}\overline{J}_{(\alpha n-1)/2}[\gamma(x-z)]\overline{J}_{(\alpha-5)/2}(\gamma z)dz.\ (16)$$

(16) dagi $\overline{J}_\nu(x)$ funksiyalarni ularning (4) ifodasi bilan almashtirsak, quyidagiga ega bo‘lamiz:

$$\overline{J}_{(\alpha n-1)/2}[\gamma(x-z)]\overline{J}_{(\alpha-5)/2}[\gamma(z)]=\sum_{m=0}^{\infty}\frac{(-1)^m(\gamma/2)^{2m}(x-z)^{2m}}{m!((\alpha n+1)/2)_m}\sum_{m=0}^{\infty}\frac{(-1)^k(\gamma/2)^{2k}z^{2k}}{k!((\alpha-3)/2)_k}.$$

Bu yerdan yaqinlashuvchi qatorlarni ko‘paytirishning Koshi qoidasini qo‘llab, quyidagi natijaga kelamiz:

$$\overline{J}_{(\alpha n-1)/2}[\gamma(x-z)]\overline{J}_{(\alpha-5)/2}[\gamma(z)]=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{(-1)^k(\gamma/2)^{2k}(x-z)^{2k}}{k!((\alpha n+1)/2)_k}\frac{(-1)^{m-k}(\gamma/2)^{2m-2k}(z)^{2m-2k}}{(m-k)!((\alpha-3)/2)_{m-k}}=$$

$$=\sum_{m=0}^{\infty}(-1)^m(\gamma/2)^{2m}\sum_{k=0}^{m}\frac{z^{2m-2k}(x-z)^{2k}}{k!(m-k)!((\alpha n+1)/2)_k((\alpha-3)/2)_{m-k}}.$$

Olingan natijani (16) tenglikka qo‘yib, integral va yig‘indining tartibini o‘zgartirsak,

$$I_{0x}^{\alpha n,\gamma}\{x^{\alpha-4}\overline{J}_{(\alpha-5)/2}(\gamma x)\}=\sum_{m=0}^{\infty}\frac{(-1)^m(\gamma/2)^{2m}}{\Gamma(\alpha n)}\times$$

$$\times\sum_{k=0}^{m}\frac{1}{k!(m-k)!((\alpha n+1)/2)_k((\alpha-1)/2)_{m-k}}\int_0^x z^{2m+\alpha-2k-4}(x-z)^{\alpha n+2k-1}dz. \quad (17)$$

Ichki integralda integrallash o‘zgaruvchisini $t=xs$ formula bo‘yicha almashtirish bajarib, ba’zi hisoblashlardan so‘ng, quyidagi natijaga ega bo‘lamiz:

$$\int_0^x z^{2m+\alpha-2k-4}(x-z)^{\alpha n+2k-1}dz=x^{\alpha n+2m+\alpha-4}\Gamma(2m-2k+\alpha-3)\Gamma(\alpha n+2k)/\Gamma(\alpha n+2m+\alpha-3).$$

Hosil bo‘lgan natijani (17) ga qo‘yib va (12) formulalarni ketma-ket qo‘llasak, ushbu tenglik hosil bo‘ladi:

$$I_{0x}^{\alpha n,\gamma}\{x^{\alpha-4}\overline{J}_{(\alpha-5)/2}(\gamma x)\}=$$

$$=\Gamma(\alpha-3)\sum_{m=0}^{\infty}\frac{(-1)^m(\gamma/2)^{2m}x^{\alpha n+2m+\alpha-4}}{\Gamma(\alpha n+2m+\alpha-3)}\sum_{k=0}^{m}\frac{(\alpha n)_{2k}(\alpha-3)_{2m-2k}}{k!(m-k)!((\alpha n+1)/2)_k((\alpha-1)/2)_{m-k}}=$$

$$=\Gamma(\alpha-3)\sum_{m=0}^{\infty}\frac{(-1)^m(\gamma/2)^{2m}2^{2m}x^{\alpha n+2m+\alpha-4}}{\Gamma(\alpha n+2m+\alpha-3)}\sum_{k=0}^{m}\frac{(\alpha n/2)_k((\alpha-2)/2)_{m-k}}{k!(m-k)!}.$$

Ushbu

$$\sum_{k=0}^{m}\frac{(\delta)_k(\gamma)_{m-k}}{k!(m-k)!}=\frac{(\delta+\gamma)_m}{m!}, \quad (18)$$

ma’lum formulani e’tiborga olsak, oxirgi tenglik ushbu ko‘rinishni oladi:

$$I_{0x}^{\alpha n,\gamma}\{x^{\alpha-4}\overline{J}_{(\alpha-5)/2}(\gamma x)\}=\Gamma(\alpha-1)\sum_{m=0}^{\infty}\frac{(-1)^m\gamma^{2m}((\alpha n+\alpha-2)/2)_m}{m!\Gamma(\alpha n+2m+\alpha-3)}.$$

Bu yerdan $\Gamma(\alpha n+2m+\alpha-3)$ uchun (12) formulalarni qo‘llab, oxirgi ifodani ushbu ko‘rinishga keltiramiz:

$$I_{0x}^{\alpha n,\gamma}\{x^{\alpha-4}\overline{J}_{(\alpha-5)/2}(\gamma x)\}=\frac{\Gamma(\alpha-3)}{\Gamma(\alpha n+\alpha-3)}\sum_{m=0}^{\infty}\frac{(-1)^m(\gamma/2)^{2m}x^{\alpha n+2m+\alpha-4}}{m!((\alpha n+\alpha-3)/2)_m}.$$

Oxirgidan (4) tenglikka asosan ushbu

$$I_{0x}^{\alpha n,\gamma}\{x^{\alpha-4}\overline{J}_{(\alpha-5)/2}(\gamma x)\}=\frac{x^{\alpha n+\alpha-4}\Gamma(\alpha-3)}{\Gamma(\alpha n+\alpha-3)}\overline{J}_{((\alpha n+\alpha-5)/2)}(\gamma x) \quad (19)$$

natijaga ega bo‘lamiz.

Yuqoridagiga o‘xshash amallar bajarib, ushbu

$$I_{0x}^{\alpha n,\gamma}\{x^{\alpha-3}\bar{J}_{(\alpha-3)/2}(\gamma x)\}=\frac{x^{\alpha n+\alpha-3}\Gamma(\alpha-2)}{\Gamma(\alpha n+\alpha-2)}\bar{J}_{((\alpha n+\alpha-3)/2)}(\gamma x) \quad (20)$$

$$I_{0x}^{\alpha n,\gamma}\{x^{\alpha-2}\bar{J}_{(\alpha-3)/2}(\gamma x)\}=\frac{x^{\alpha n+\alpha-2}\Gamma(\alpha-1)}{\Gamma(\alpha n+\alpha-1)}\bar{J}_{((\alpha n+\alpha-3)/2)}(\gamma x) \quad (21)$$

$$I_{0x}^{\alpha n,\gamma}\{x^{\alpha-1}\bar{J}_{(\alpha-1)/2}(\gamma x)\}=\frac{x^{\alpha n+\alpha-1}\Gamma(\alpha)}{\Gamma(\alpha n+\alpha)}\bar{J}_{((\alpha n+\alpha-1)/2)}(\gamma x) \quad (22)$$

tengliklar o‘rinli ekanligini ko‘rsatish mumkin.

(19),(20) ,(21) va (22) larni (14) tenglikka qo‘yib va (3) operatorning yoyilmasini e’tiborga olib, (6) formulaga ega bo‘lamiz.

4. Asosiy yechim

Endi (6) formula bilan aniqlangan $y(x)$ funksiyani (1) tenglamani va (5) shartni qanoatlantirishini ko‘rsatamiz. Shu maqsadda uni ushbu

$$y(x)=y_1(x)+y_2(x)+y_3(x)+y_4(x)+y_5(x)$$

ko‘rinishida yozib olamiz, bu yerda

$$y_1(x)=A_1x^{\alpha-4}\mathrm{E}_{\alpha,\alpha-3,(\alpha-5)/2}\left[-\lambda x^{\alpha};\gamma x\right],\quad y_2(x)=A_2x^{\alpha-3}\mathrm{E}_{\alpha,\alpha-2,(\alpha-3)/2}\left[-\lambda x^{\alpha};\gamma x\right],$$

$$y_3(x)=A_3x^{\alpha-2}\mathrm{E}_{\alpha,\alpha-1,(\alpha-3)/2}[-\lambda x^{\alpha};\gamma x],\quad y_4(x)=A_4x^{\alpha-1}\mathrm{E}_{\alpha,\alpha,(\alpha-1)/2}[-\lambda x^{\alpha};\gamma x],$$

$$y_5(x)=\int_0^x(x-z)^{\alpha-1}\mathrm{E}_{\alpha,\alpha,(\alpha-1)/2}\left[-\lambda(x-z)^{\alpha};\gamma(x-z)\right]f(z)dz$$

(2) formulani e’tiborga olib, dastlab $I_{0x}^{4-\alpha,\gamma}y_1(x)$ ni hisoblaymiz:

$$I_{0x}^{4-\alpha,\gamma}y_1(x)=\frac{1}{\Gamma(4-\alpha)}\int_0^x(x-z)^{3-\alpha}\bar{J}_{((3-\alpha)/2)}\left[\gamma(x-z)\right]y_1(z)dz=$$

$$=\frac{A_1}{\Gamma(4-\alpha)}\int_0^x(x-z)^{3-\alpha}z^{\alpha-4}\bar{J}_{((3-\alpha)/2)}\left[\gamma(x-z)\right]\mathrm{E}_{\alpha,\alpha-3,((\alpha-5)/2)}\left[-\lambda z^{\alpha};\gamma z\right]dz=$$

$$=A_1\sum_{n=0}^{+\infty}\frac{(-\lambda)^n}{\Gamma(\alpha n+\alpha-3)}\frac{1}{\Gamma(4-\alpha)}\int_0^x(x-z)^{3-\alpha}z^{\alpha n+\alpha-4}\bar{J}_{(3-\alpha)/2}\left[\gamma(x-z)\right]\bar{J}_{((\alpha n+\alpha-5)/2)}(\gamma z)dz.$$

Quyidagi belgilashni kiritaylik:

$$H(\alpha,n,\gamma;x)=$$

$$=\frac{1}{\Gamma(\alpha n+\alpha-3)\Gamma(4-\alpha)}\int_0^x(x-z)^{3-\alpha}z^{\alpha n+\alpha-4}\bar{J}_{(3-\alpha)/2}\left[\gamma(x-z)\right]\bar{J}_{(\alpha n+\alpha-5)/2}(\gamma z)dz. \quad (23)$$

U holda oxirgi tenglik quyidagicha yoziladi:

$$I_{0x}^{4-\alpha,\gamma}y_1(x)=A_1\sum_{n=0}^{\infty}(-\lambda)^nH(\alpha,n,\gamma;x) \quad (24)$$

Endi $H(\alpha,n,\gamma;x)$ funksiyani soddalashtiraylik. Shu maqsadda $\bar{J}_{\nu}(x)$ ko‘rinishini uning (4) ifodasi bilan almashtirib, qatorlarni ko‘paytirishning Koshi qoidasini qo‘llab, quyidagini hosil qilamiz:

$$\bar{J}_{(3-\alpha)/2}\left[\gamma(x-z)\right]\bar{J}_{(\alpha n+\alpha-5)/2}(\gamma z)=$$

$$=\sum_{m=0}^{\infty}(-1)^m(\gamma/2)^{2m}\sum_{k=0}^{m}\frac{z^{2k}(x-z)^{2m-2k}}{k!(m-k)!((5-\alpha)/2)_{m-k}((\alpha n+\alpha-3)/2)_k}.$$

Oxirgini e'tiborga olib, (24) dan quyidagini hosil qilamiz

$$H(\alpha,n,\gamma;x)=$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m(\gamma/2)^{2m}}{\Gamma(\alpha n+\alpha-3)\Gamma(4-\alpha)}\sum_{k=0}^{m}\frac{H_1(\alpha,n,m,k;x)}{k!(m-k)!((5-\alpha)/2)_{m-k}((\alpha n+\alpha-3)/2)_k},$$

bu yerda

$$H_1(\alpha,n,m,k;x)=\int_0^x z^{\alpha n+\alpha+2k-4}(x-z)^{3+2m-2k-\alpha}\,dz.$$

Ko'rsatish qiyin emaski, $H_1(\alpha,n,m,k;x)$ funksiya uchun ushbu tenglik o'rinli:

$$H_1(\alpha,n,m,k;x)=x^{\alpha n+2m}\frac{\Gamma(2m-2k+4-\alpha)\Gamma(\alpha n+\alpha+2k-3)}{\Gamma(\alpha n+2m+1)}.$$

Bu natijani (25) ga qo'yib, $\Gamma(\alpha n+\alpha+2k-3)$ va $\Gamma(2m-2k+4-\alpha)$ lar uchun (11) formulani ketma-ket qo'llasak , quyidagi natijaga ega bo'lamiz:

$$H(\alpha,n,\gamma;x)=$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m(\gamma/2)^{2m}x^{2m+\alpha n}}{\Gamma(\alpha n+2m+1)}\sum_{k=0}^{m}\frac{(4-\alpha)_{2m-2k}(\alpha n+\alpha-3)_{2k}}{k!(m-k)!((\alpha n+\alpha-3)/2)_k((5-\alpha)/2)_{m-k}}=$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m\gamma^{2m}x^{2m+\alpha n}}{\Gamma(\alpha n+2m+1)}\sum_{k=0}^{m}\frac{((4-\alpha)/2)_{m-k}((\alpha n+\alpha-2)/2)_k}{k!(m-k)!}.$$

Agar (18) tenglikni hisobga olsak, $H(\alpha,n,\gamma;x)$ funksiya quyidagi ko'rinishni oladi:

$$H(\alpha,n,\gamma;x)=\sum_{m=0}^{\infty}\frac{(-1)^m\gamma^{2m}((\alpha n+\alpha)/2)_k x^{2m+\alpha n}}{m!\Gamma(\alpha n+2m+1)}. \quad (25)$$

$H(\alpha,n,\gamma;x)$ funksiyaning (25) ifodasini (24) qo'yib, ushbu

$$I_{0x}^{4-\alpha,\gamma}y_1(x)=A_1\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-\lambda)^n(-1)^m\gamma^{2m}((\alpha n+\alpha)/2)_m x^{2m+\alpha n}}{m!\Gamma(\alpha n+2m+1)}$$

tenglikni hosil qilamiz.

$\Gamma(\alpha n+2m+1)$ uchun (12) formulalarni ketma-ket qo'llab, va (7) ni e'tiborga olgan holda, ushbu

$$I_{0x}^{4-\alpha,\gamma}y_1(x)=A_1\mathrm{E}_{\alpha,1,-1/2}\left[-\lambda x^{\alpha};\gamma x\right] \quad (26)$$

natijaga ega bo'lamiz. .

Yuqoridagi kabi ushbu

$$I_{0x}^{4-\alpha,\gamma}y_2(x)=A_2x\mathrm{E}_{\alpha,2,1/2}\left[-\lambda x^{\alpha};\gamma x\right], \quad (27)$$

$$I_{0x}^{4-\alpha,\gamma}y_3(x)=A_3x^2\mathrm{E}_{\alpha,3,1/2}\left[-\lambda x^{\alpha};\gamma x\right], \quad (28)$$

$$I_{0x}^{4-\alpha,\gamma}y_4(x)=A_4x^3\mathrm{E}_{\alpha,4,3/2}\left[-\lambda x^{\alpha};\gamma x\right], \quad (29)$$

$$I_{0x}^{4-\alpha,\gamma}y_5(x)=\int_0^x (x-z)^3 \mathrm{E}_{\alpha,4,3/2}\left[-\lambda(x-z)^\alpha;\gamma(x-z)\right]f(z)dz, \qquad (30)$$

tengliklarning o‘rinli ekanligi ham ko‘rsatish mumkin. .

(27), (28), (29), (30), (31) tengliklarni e’tiborga olib, $I_{0x}^{4-\alpha,\gamma}y(x)$ funksiya uchun ushbu ifodani hosil qilamiz:

$$I_{0x}^{4-\alpha,\gamma}y(x)=A_1\mathrm{E}_{\alpha,1,-1/2}\left[-\lambda x^\alpha;\gamma x\right]+A_2x\mathrm{E}_{\alpha,2,1/2}\left[-\lambda x^\alpha;\gamma x\right]+A_3x^2\mathrm{E}_{\alpha,3,1/2}\left[-\lambda x^\alpha;\gamma x\right]+$$

$$+A_4x^3\mathrm{E}_{\alpha,4,3/2}\left[-\lambda x^\alpha;\gamma x\right]+\int_0^x (x-z)^3 \mathrm{E}_{\alpha,4,3/2}\left[-\lambda(x-z)^\alpha;\gamma(x-z)\right]f(z)dz. \quad (31)$$

$D_{0x}^{\alpha,\gamma}y(x)=\left(\frac{d^2}{dx^2}+\gamma^2\right)^2 I_{0x}^{4-\alpha,\gamma}y(x)$ ekanligini va (8), (9) hamda $\mathrm{E}_{\alpha,\beta,\theta}[0;0]=\frac{1}{\Gamma(\beta)}$ tengliklarni e’tiborga olib, ushbu

$$\left(\frac{d^2}{dx^2}+\gamma^2\right)^2 I_{0x}^{4-\alpha,\gamma}y(x)=$$

$$A_1\left(\frac{d^2}{dx^2}+\gamma^2\right)^2 \mathrm{E}_{\alpha,1,-1/2}\left[-\lambda x^\alpha;\gamma x\right]+A_2\left(\frac{d^2}{dx^2}+\gamma^2\right)^2 x\mathrm{E}_{\alpha,2,1/2}\left[-\lambda x^\alpha;\gamma x\right]+$$

$$+A_3\left(\frac{d^2}{dx^2}+\gamma^2\right)^2 x^2\mathrm{E}_{\alpha,3,1/2}\left[-\lambda x^\alpha;\gamma x\right]+A_4\left(\frac{d^2}{dx^2}+\gamma^2\right)^2 x^3\mathrm{E}_{\alpha,4,3/2}\left[-\lambda x^\alpha;\gamma x\right]+$$

$$+\left(\frac{d^2}{dx^2}+\gamma^2\right)^2\int_0^x (x-z)^3 \mathrm{E}_{\alpha,4,3/2}\left[-\lambda(x-z)^\alpha;\gamma(x-z)\right]f(z)dz.$$

tenglikni hosil qilamiz. Oxirgi tenglikka (8) va (9) formulalarni ketma-ket qo‘llab, $\left(\frac{d^2}{dx^2}+\gamma^2\right)\mathrm{E}_{\alpha,1,-1/2}\left[-\lambda x^\alpha;\gamma x\right]$ni quyidagi ko‘rinishda yozishimiz mumkin:

$$\left(\frac{d^2}{dx^2}+\gamma^2\right)\mathrm{E}_{\alpha,1,-1/2}\left[-\lambda x^\alpha;\gamma x\right]=-\lambda x^{\alpha-2}\mathrm{E}_{\alpha,\alpha-1,(\alpha-3)/2}\left[-\lambda x^\alpha;\gamma x\right].$$

Endi (7) va (4) tengliklarga asosan $\left(\frac{d^2}{dx^2}+\gamma^2\right)x^{\alpha-2}\mathrm{E}_{\alpha,\alpha-1,(\alpha-3)/2}\left[-\lambda x^\alpha;\gamma x\right]$ funksiyani quyidagi ko‘rinishda yozib olamiz:

$$\left(\frac{d^2}{dx^2}+\gamma^2\right)x^{\alpha-2}\mathrm{E}_{\alpha,\alpha-1,(\alpha-3)/2}\left[-\lambda x^\alpha;\gamma x\right]=\left(\frac{d^2}{dx^2}+\gamma^2\right)\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\Gamma(\alpha n+\alpha-1)}\sum_{m=0}^{\infty}\frac{(-1)^m(\gamma/2)^{2m}x^{2m+\alpha n+\alpha-2}}{m!\left((\alpha n+\alpha-1)/2\right)_m}.$$

Oxirgi tenglikning o‘ng tarafini hadma-had ko‘paytirib, kerakli hosilani hisoblaymiz va $\left((\alpha n+\alpha-1)/2\right)_m$ ifoda uchun (12) formulani qo‘llasab, tenglamani quyidagi ko‘rinishga keltiramiz:

$$\left(\frac{d^2}{dx^2}+\gamma^2\right)x^{\alpha-2}\mathrm{E}_{\alpha,\alpha-1,(\alpha-3)/2}\left[-\lambda x^\alpha;\gamma x\right]=x^{\alpha-4}\sum_{n=0}^{\infty}\frac{(-\lambda)^n x^{\alpha n}}{\Gamma(\alpha n+\alpha-3)}\sum_{m=0}^{\infty}\frac{(-1)^m(\gamma/2)^{2m}x^{2m}}{m!\left((\alpha n+\alpha-3)/2\right)_m}.$$

U holda, (7) tenglikka asosan, ushbu

$$\left(\frac{d^2}{dx^2}+\gamma^2\right)^2 I_{0x}^{4-\alpha,\gamma} y_1(x) = -\lambda A_1 x^{\alpha-4} \mathrm{E}_{\alpha,\alpha-3,(\alpha-5)/2}\left[-\lambda x^{\alpha};\gamma x\right], \ (32)$$

natija kelib chiqadi.

Yuqoridagi kabi hisoblashlar bilan $\left(\frac{d^2}{dx^2}+\gamma^2\right)^2 I_{0x}^{4-\alpha,\gamma} y(x)$ tenglikning boshqa hadlarini ham soddalashtirish mumkin. Shunday qilib, (31) va (32) tengliklardan shunday natijaga kelamiz:

$$D_{ox}^{\alpha,\gamma} y(x) = \left(\frac{d^2}{dx^2}+\gamma^2\right)^2 I_{0x}^{4-\alpha,\gamma} y(x) = -\lambda A_1 x^{\alpha-4} \mathrm{E}_{\alpha,\alpha-3,(\alpha-5)/2}\left[-\lambda x^{\alpha};\gamma x\right] -$$

$$-\lambda A_2 x^{\alpha-3} \mathrm{E}_{\alpha,\alpha-2,(\alpha-3)/2}\left[-\lambda x^{\alpha};\gamma x\right] - \lambda A_3 x^{\alpha-2} \mathrm{E}_{\alpha,\alpha-1,(\alpha-3)/2}[-\lambda x^{\alpha};\gamma x] - \lambda A_4 x^{\alpha-1} \mathrm{E}_{\alpha,\alpha,(\alpha-1)/2}[-\lambda x^{\alpha};\gamma x] +$$

$$+ f(x) - \lambda \int_0^x (x-z)^{\alpha-1} \mathrm{E}_{\alpha,\alpha,(\alpha-1)/2}\left[-\lambda (x-z)^{\alpha};\gamma (x-z)\right] f(z)\,dz. \ (33)$$

Topilgan (33) hamda (6) tengliklarni taqqoslab, (6) formula bilan aniqlangan $y(x)$ funksiya (1) tenglamani qanoatlantiradi degan xulosaga kelamiz.

Endi (5) shartni qanoatlantirishini ko‘rsatamiz.

(31) tenglikni qaraylik, $f(x) = x^{-p} f_1(x)$ bo‘lganligi sababli, darhol $\lim_{x\to 0} I_{0x}^{4-\alpha,\gamma} y(x) = A_1$ kelib chiqadi.

$\frac{d}{dx} I_{0x}^{4-\alpha,\gamma} y(x)$ va $\left(\frac{d^2}{dx^2}+\gamma^2\right) I_{0x}^{4-\alpha,\gamma} y(x)$ ifodalardan , $f(x) = x^{-p} f_1(x)$ bo‘lganligi sababli mos holda $\lim_{x\to 0} \frac{d}{dx} I_{0x}^{4-\alpha,\gamma} y(x) = A_2$ va $\lim_{x\to 0}\left(\frac{d^2}{dx^2}+\gamma^2\right) I_{0x}^{4-\alpha,\gamma} y(x) = A_3$ ekanligi kelib chiqadi.

(31) tenglikdan, (8) va (9) formulalarga binoan, quyidagi yakuniy natijaga ega bo‘lamiz:

$$\left(\frac{d^2}{dx^2}+\gamma^2\right)^2 I_{0x}^{4-\alpha,\gamma} y(x) = -\lambda A_1 x^{\alpha-4} \mathrm{E}_{\alpha,\alpha-3,(\alpha-5)/2}\left[-\lambda x^{\alpha};\gamma x\right] -$$

$$-\lambda A_2 x^{\alpha-3} \mathrm{E}_{\alpha,\alpha-2,(\alpha-3)/2}\left[-\lambda x^{\alpha};\gamma x\right] - \lambda A_3 x^{\alpha-2} \mathrm{E}_{\alpha,\alpha-1,(\alpha-3)/2}[-\lambda x^{\alpha};\gamma x] - \lambda A_4 x^{\alpha-1} \mathrm{E}_{\alpha,\alpha,(\alpha-1)/2}[-\lambda x^{\alpha};\gamma x] +$$

$$+ f(x) - \lambda \int_0^x (x-z)^{\alpha-1} \mathrm{E}_{\alpha,\alpha,(\alpha-1)/2}\left[-\lambda (x-z)^{\alpha};\gamma (x-z)\right] f(z)\,dz,$$

Demak, $E_{\alpha,\beta,\theta}[0;0] = \Gamma^{-1}(\beta)$, $3 < \alpha < 4$ va $f(x) = x^{-p} f_1(x)$ ekanligidan,

$\lim_{x\to 0} \frac{d}{dx}\left(\frac{d^2}{dx^2}+\gamma^2\right) I_{0x}^{4-\alpha,\gamma} y(x) = A_4$ ekanligi kelib chiqadi.

Teorema isbotlandi.

Foydalanilgan adabiyotlar.

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