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Uch O'lchovli Lavrentev-Bitsadze Tenglamasi Uchun Trikomi Masalasi

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Annotatsiya: Mazkur ishda uch o'lchovli Lavrentev-Bitsadze tenglamasi uchun chegaralangan sohada Trikomi tipidagi masala o'rganilgan. Masala yechimiming mavjudligi va yagonaligi spektral analiz usuli yordamida isbotlangan.

Kalit so'zlar: Uch o'chovli Lavrentev-Bitsadze tenglamasi, Trikomi masalasi, spektral analiz usuli.

I. Kirish. Masalaning qo'yilishi.

Tekislikda aralash elliptiko-giperbolik tipdagi

$$yu_{xx} + u_{yy} = 0 \qquad (0)$$

tenglama uchun birinchi fundamental tadqiqotlarni 1923-yilda Italyan matematigi Franchesko Trikomi olib brogan [1]. U hozirgi vaqtda uning nomi bilan ataluvchi Trikomi masalasini qo'ygan va tadqiq qilgan. Hozirgi paytda tekislikda bu tipdagi masalalar turli tenglamalar uchun anchayin rivojlangan [2-3]. Lekin ko'plab masalalarda masala yechimi oshkor ko'rinishda topilmaydi.

Mazkur ishda esa uch o'lchovli fazoda masala yechimi spektral analiz usuli yordamida oshkor ko'rinishda topiladi.

Aytaylik $\Omega = \{(x,y,z):(x,y)\in\Delta, z\in(0,c)\}$ bo'lsin, bu yerda $\Delta-$ xOy tekisligining bir bog'lamli sohasi bo'lib, $y\ge 0$ bo'lganda $\bar{\sigma}_0 = \left\{(x,y): x^2+y^2=1, x\ge 0, y\ge 0\right\}$ yoy va $\overline{OM} = \left\{(x,y): x=0, 0\le y\le 1\right\}$ kesma bilan $y\le 0$ da esa $\overline{OQ} = \left\{(x,y): x+y=0, 0\le x\le 1/2\right\}$ va $\overline{QP} = \left\{(x,y): x-y=1, 1/2\le x\le 1\right\}$ kesmalar bilan chegaralangan soha bo'lsin; $O=O(0,0)$, $M=M(0,1)$, $P=P(1,0)$, $Q=Q(1/2,-1/2)$.

Quyidagi belgilashlarni kiritamiz: $\Omega_0 = \Omega\cap(y>0)$, $\Omega_1 = \Omega\cap(y<0)$; $\Delta_0 = \Delta\cap(y>0)$, $\Delta_1 = \Delta\cap(y<0)$; $\bar{S}_0 = \left\{(x,y,z): \bar{\sigma}_0\times[0,c]\right\}$,

$\bar{S}_1 = \left\{(x,y,z): \overline{OM}\times[0,c]\right\}$, $\bar{S}_2 = \left\{(x,y,z): \overline{OQ}\times[0,c]\right\}$;

$\bar{S}_3 = \left\{(x,y,z): \bar{\Omega}\cap(z=0)\right\}$, $\bar{S}_4 = \left\{(x,y,z): \bar{\Omega}\cap(z=c)\right\}$.

Ω sohada quyidagi

$$U_{xx} + (\text{sgn}y)U_{yy} + U_{zz} = 0 \qquad (1)$$

tenglamani qaraymiz.

[1,2] Farg'ona davlat universiteti

Ω sohada (1) tenglama aralash tipga tegishli bo‘lib, jumladan Ω_0 -sohada elliptik tipga, Ω_1 sohada esa giperbolik tipga tegishlidir.

(1) tenglama uchun Ω sohada quyidagi masalani tadqiq qilamiz.

T (Trikomi) masala. Ω sohada (1) tenglamani va quyidagi:

$$U(x,y,z)\in C(\bar{\Omega})\cap C^{2,2,2}_{x,y,z}(\Omega_0\cup\Omega_1),U_x,U_y,U_z\in C(\bar{\Omega}_0); \quad (2)$$

$$U(x,y,z)\big|_{\bar{S}_0}=F(x,y,z); \quad (3)$$

$$U(x,y,z)\big|_{\bar{S}_1}=0,\, U(x,y,z)\big|_{\bar{S}_2}=0, \quad (4)$$

$$U(x,y,z)\big|_{\bar{S}_3}=0,\, U(x,y,z)\big|_{\bar{S}_4}=0, \quad (5)$$

$$\lim_{y\to -0}U_y(x,y,z)=\lim_{y\to +0}U_y(x,y,z), \quad x\in(0,1),\ z\in(0,c), \quad (6)$$

shartlarni qanoatlantiruvchi $U(x,y,z)$ funksiya topilsin, bu yerda $F(x,y,z)$ – berilgan funksiya.

II. Giperbolik va elliptik sohalarda (1) tenglamaning xususiy yechimlarini topish

Masala yechimini *Furye* usuli ya’ni

$$U(x,y,z)=u(x,y)Z(z) \quad (7)$$

o‘zgaruvchilari ajraladigan funksiya sifatida qidiramiz, bu yerda $u(x,y)$ va $Z(z)$-noma’lum funksiyalar.

Biz qidirayotgan (7) yechim, (1) tenglamani va (2)-(6) shartlarni qanoatlantirishi kerak. Shu sababdan funksiyadan kerakli hosilalarni olib (1) tenglamaga qo‘yib quyidagi tenglamani hosil qilamiz:

$$u_{xx}(x,y)Z(z)+\operatorname{sgn} y u_{yy}(x,y)Z(z)+u(x,y)Z''(z)=0, \quad (8)$$

o‘z navbatida bu ifodani $u(x,y)Z(z)$ ga bo‘lib quyidagi

$$u_{xx}(x,y)+\operatorname{sgn} y u_{yy}(x,y)-\lambda u(x,y)=0 \quad (9)$$

$$u(0,y)=0,\ y\in[0,1], \quad u(x,-x)=0,\ x\in[0,1/2]; \quad (10)$$

ikkinchi tartibli ikki o‘zgaruvchili xususiy hosilali differensial tenglama va

$$Z''(z)+\lambda Z(z)=0, \quad z\in[0,c] \quad (11)$$

ikkinchi tartibli oddiy differensial tenglamani hosil qilamiz, bunda $\lambda=const$.

(7) ifoda va (4) shartlardan (11) tenglama uchun $Z(0)=0$ va $Z(c)=0$ chegaraviy shartlar kelib chiqadi.

(11) va $Z(0)=0$ va $Z(c)=0$ chegaraviy shartlardan iborat masala xos qiymat haqidagi *Shturm-Liuvill* masalasidir. Bu masalaning xos qiymatlari quyidagicha

$$\lambda_m=\left(\frac{\pi m}{c}\right)^2,\ m\in N \quad (12)$$

bu qiymatlarga mos trivial bo‘lmagan (aynan nolga teng bo‘lmagan) ortonormallashgan xos funksiyalari esa

$$Z_m(z)=\sqrt{\frac{2}{c}}\sin\frac{\pi m}{c}z,\ m\in N \qquad (13)$$

ko‘rinishida bo‘ladi.

E.I. Moiseevning [4] ishida quyidagi tasdiq keltirilgan.

1-tasdiq. Aytaylik $p\in(1,+\infty)$ va $-\frac{1}{p}<\frac{\gamma}{\pi}<2-\frac{1}{p}$ bo‘lsin. U holda

$$\left\{\sin\left[(k+\beta/2)\theta+\gamma/2\right]\right\}_{k=1}^{\infty},$$

funksiyalar sistemasi faqat va faqat

$$\frac{1}{p}>\frac{\gamma}{\pi}+\beta>\frac{1}{p}-2$$

tengsizlik o‘rinli bo‘lganda $L_p(0,\pi)$ fazoda bazis tashkil qiladi ($p=2$ da Riss bazisini tashkil qiladi), bu yerda $\theta\in[0,\pi]$, β va γ - ixtiyoriy haqiqiy sonlar.

Agar bu tasdiqda $k=m$, $\beta=0$, $\theta=\pi z/c$, $\gamma=0$ va $p=2$ desak, $\left\{\sin\frac{\pi mz}{c}\right\}_{m=1}^{\infty}$ sistema hosil bo‘ladi. Bu aynan biz tadqiq qilayotgan masalaning xos funsiyalari sistemasidir. Xulosa o‘rnida shuni aytish mumkinki, $\left\{\sin\frac{\pi mz}{c}\right\}_{m=1}^{\infty}$ funksiyalar sistemasi $L_2(0,c)$ fazoda Riss bazisini tashkil qiladi. Ya’ni shu fazodan olingan ixtiyoriy funksiyani mazkur funksiyalar sistemasi yordamida qatorga yoyish mumkinligini bildiradi.

Endi {(9),(10)} masalani $\lambda=\lambda_m$ bo‘lgan holda Δ_1 sohada ko‘rib chiqamiz:

$$u_{xx}-u_{yy}-\lambda_m u=0,\quad (x,y)\in\Delta_1, \qquad (14)$$

$$u(x,-x)=0,\quad x\in[0,1/2]. \qquad (15)$$

{(14),(15)} masalaning yechimini quyidagicha qidiramiz:

$$u(x,y)=X(\xi)Y(\eta),\ \text{bu yerda}\ \xi=\sqrt{x^2-y^2},\ \eta=x^2/\xi^2. \qquad (16)$$

U holda $X(\xi)$ va $Y(\eta)$ funksiyalar uchun quyidagi shartlar $X(0)=0$, $\left|\lim_{\eta\to+\infty}Y(\eta)\right|<+\infty$ va ba’zi hisoblanadigan xususiy hosilalar bilan

$$u_{xx}=X''(\xi)Y(\eta)\xi_x^2+2X'(\xi)Y'(\eta)\xi_x\eta_x+X(\xi)Y''(\eta)\eta_x^2+X'(\xi)Y(\eta)\xi_{xx}+X(\xi)Y'(\eta)\eta_{xx}$$

$$u_{yy}=X''(\xi)Y(\eta)\xi_y^2+2X'(\xi)Y'(\eta)\xi_y\eta_y+X(\xi)Y''(\eta)\eta_y^2+X'(\xi)Y(\eta)\xi_{yy}+X(\xi)Y'(\eta)\eta_{yy}$$

bu yerda,

$$\xi_x=\frac{x}{\sqrt{x^2-y^2}},\quad \xi_{xx}=\frac{-y^2}{\sqrt{(x^2-y^2)^3}},\quad \eta_x=\frac{-2xy^2}{(x^2-y^2)^2},\quad \eta_{xx}=\frac{2y^2(3x^2+y^2)}{(x^2-y^2)^3},$$

$$\xi_y = \frac{-y}{\sqrt{x^2-y^2}},\ \xi_{yy} = \frac{-x^2}{\sqrt{(x^2-y^2)^3}},\ \eta_y = \frac{2x^2y}{(x^2-y^2)^2},\ \eta_{yy} = \frac{2x^2(x^2+3y^2)}{(x^2-y^2)^3},$$

(14) tenglama quyidagi

$$\xi^2 X''(\xi) + \xi X'(\xi) - \left[\lambda_m \xi^2 + \mu\right] X(\xi) = 0,\ \xi > 0; \qquad (17)$$

$$\eta(1-\eta)Y''(\eta) + [1/2-\eta]Y'(\eta) + \frac{1}{4}\mu Y(\eta) = 0,\ \eta > 1, \qquad (18)$$

tenglamalar kelib chiqadi, bu yerda $\mu = const$.

Birinchi bo‘lib, {(17), $X(0)=0$} masalaning yechimini topamiz. (17) tenglamaning umumiy yechimi quyidagi ko‘rinishda aniqlanadi:

$$X(\xi) = c_1 I_\omega\left(\xi\sqrt{\lambda_m}\right) + c_2 K_\omega\left(\xi\sqrt{\lambda_m}\right),\ m \in N,$$

bu yerda $I_\nu(x)$ va $K_\nu(x)$ funksiyalar mos holda Infeld va Makdonald funksiyalari [5], $\omega = \sqrt{\mu}$, $\mu > 0$.

Bu yechimni $X(0)=0$ shartga bo‘ysundiramiz. Makdonald funksiyasi $\xi = 0$ nuqtada chegaralanmaganligidan $c_2 = 0$ deb olamiz. Xulosa o‘rnida shuni aytish mumkinki, (17) tenglamaning $X(0)=0$ shartni qanoatlantiruvchi yechimi $\mu > 0$ bo‘lganda o‘zgarmas ko‘paytuvchi aniqligida quyidagi ko‘rinishga ega

$$X(\xi) = I_\omega\left(\xi\sqrt{\lambda_m}\right),\ m \in N. \qquad (19)$$

(18) esa Gaussning gipergeometrik tenglamasi hisoblanadi [6]. Uning umumiy yechimini quyidagi

$$Y(\eta) = c_3\eta^{-\omega/2} F(\omega/2, 1/2+\omega/2, 1+\omega; 1/\eta) +$$

$$c_4\eta^{\omega/2} F(-\omega/2, 1-\omega/2, 1-\omega; 1/\eta), \qquad (20)$$

formula bilan aniqlanadi, bu yerda c_3, c_4 – ixtiyoriy o‘zgarmas sonlar.

Demak $\omega > 0$, u holda (20) umumiy yechimning $\eta \to +\infty$ da chegaralangan bo‘lishi uchun $c_4 = 0$ desak quyidagi masalaning umumiy yechimiga ega bo‘lamiz:

$$Y(\eta) = c_3\eta^{-\omega/2} F(\omega/2, 1/2+\omega/2, 1+\omega; 1/\eta).$$

Bu funksiyani ma’lum bo‘lgan

$$F(a-1/2, a, 2a; \eta) = \left(\frac{1}{2} + \frac{1}{2}\sqrt{1-\eta}\right)^{1-2a},$$

formulaga [6] asosan quyidagich yozishimiz mumkin:

$$Y(\eta) = c_3 2^\omega \left(\frac{x+y}{x-y}\right)^{\omega/2}. \qquad (21)$$

Demak, $\bar{\Delta}_1$ da {(14),(15)} masalaning yechimini (16), (19), (21) larga asosan

$$u_m^-(x,y)=c_3 2^{\omega}\left(\frac{x+y}{x-y}\right)^{\omega/2} I_{\omega}\left(\sigma_m\sqrt{x^2-y^2}/c\right),\ c_1\neq 0,\ m\in N.\quad (22)$$

ko'rinishda yozib olamiz.

(22) dan quyidagilarni topamiz:

$$\begin{cases}\tau_m^-(x)=\lim\limits_{y\to -0}u_m^-(x,y)=c_3 2^{\omega}I_{\omega}(\sigma_m x/c),\ x\in[0,1];\\ \nu_m^-(x)=\lim\limits_{y\to -0}\dfrac{\partial}{\partial y}u_m^-(x,y)=c_3\omega 2^{\omega}x^{-1}I_{\omega}(\sigma_m x/c),\ x\in(0,1).\end{cases}\quad (23)$$

Endi {(9),(10)} masalani $\lambda=\lambda_m$ bo'lgan holda Δ_0 sohada ko'rib chiqamiz, ya'ni quyidagi masalani tadqiq qilamiz:

$$u_{xx}+u_{yy}-\lambda_m u=0,\quad (x,y)\in\Delta_0,\quad (24)$$

$$u(0,y)=0,\quad y\in[0,1].\quad (25)$$

Masala yechimini o'zgaruvchilari ajraladigan funksiya sifatida qidiramiz:

$$u(x,y)=Q(\rho)S(\varphi),\quad (26)$$

bu yerda $\rho=\sqrt{x^2+y^2},\ \varphi=arctg(y/x).$

(26) ga asosan xususiy hosilalarni hisoblaymiz:

$$u_{xx}=Q''(\rho)S(\varphi)\rho_x^2+2Q'(\rho)S'(\varphi)\rho_x\varphi_x+Q(\rho)S''(\varphi)\varphi_x^2+Q'(\rho)S(\varphi)\rho_{xx}+Q(\rho)S'(\varphi)\varphi_{xx}$$

$$u_{yy}=Q''(\rho)S(\varphi)\rho_y^2+2Q'(\rho)S'(\varphi)\rho_y\varphi_y+Q(\rho)S''(\varphi)\varphi_y^2+Q'(\rho)S(\varphi)\rho_{yy}+Q(\rho)S'(\varphi)\varphi_{yy}$$

bu yerda,

$$\rho_x=\frac{x}{\sqrt{x^2+y^2}},\quad \rho_{xx}=\frac{y^2}{\sqrt{(x^2+y^2)^3}},\quad \varphi_x=\frac{-y}{x^2+y^2},\quad \varphi_{xx}=\frac{2xy}{(x^2+y^2)^2},$$

$$\rho_y=\frac{y}{\sqrt{x^2+y^2}},\quad \rho_{yy}=\frac{x^2}{\sqrt{(x^2+y^2)^3}},\quad \varphi_y=\frac{x}{x^2+y^2},\quad \varphi_{yy}=\frac{-2xy}{(x^2+y^2)^2},$$

olingan xususiy hosilalardan va ba'zi algebraik soddalashtirishlarni amalga oshiramiz.

(26) formula {(24),(25)} masalani (25) va $u\in C(\bar{\Delta}_0)$ shartlarni e'tiborga olgan holda quyidagi xos qiymat haqidagi masalalarga ajratadi:

$$\rho^2 Q''(\rho)+\rho Q'(\rho)-\left[\lambda_m\rho^2+\tilde{\mu}\right]Q(\rho)=0,\ \rho\in(0,1),\quad (27)$$

$$|Q(0)|<+\infty;\quad (28)$$

$$S''(\varphi)+\tilde{\mu}S(\varphi)=0,\ \varphi\in(0,\pi/2),\quad (29)$$

$$S(\pi/2)=0, \qquad (30)$$

bu yerda $\tilde{\mu}=const$.

Dastlab {(27),(28)} masalani qaraylik. Bizga ma'lumki (27) tenglamaning umumiy yechimi [5]

$$Q_m(\rho)=c_5 I_{\tilde{\omega}}\left(\sqrt{\lambda_m}\rho\right)+c_6 K_{\tilde{\omega}}\left(\sqrt{\lambda_m}\rho\right),\ \rho\in[0,1], \qquad (31)$$

formula yordamida aniqlanadi, bu yerda $\tilde{\omega}=\sqrt{\tilde{\mu}},\ (\tilde{\mu}>0)$, c_5 va c_6 esa ixtiyoriy o'zgarmas sonlar.

(31) umumiy yechimda (28) shartni qanoatlantiradigan trivial bo'lmagan yechimini quyidagicha yozib olishimiz mumkin

$$Q_m(\rho)=c_5 I_{\tilde{\omega}}\left(\sqrt{\lambda}\rho\right),\ m\in N. \qquad (32)$$

Endi {(29),(30)} masalani tadqiq qilamiz. (29) tenglamaning umumiy yechimi

$$S(\varphi)=c_7\cos\sqrt{\tilde{\mu}}\varphi+c_8\sin\sqrt{\tilde{\mu}}\varphi \qquad (33)$$

formula bilan aniqlanadi, bu yerda c_7 va c_8 - ixtiyoriy o'zgarmas sonlar.

(33) funksiyani (30) shartaga buysundiramiz va $c_8=-ctg(\tilde{\omega}\pi/2)c_7$ tenglikka ega bo'lamiz (umumiylikni chegaralamay $c_7=1$ olamiz):

$$S(\varphi)=\cos(\tilde{\omega}\varphi)-ctg(\tilde{\omega}\pi/2)\sin(\tilde{\omega}\varphi). \qquad (34)$$

Demak, $\bar{\Delta}_0$ da {(24),(25)} masalaning yechimini (26), (32), (34) tengliklarga asosan

$$u_m^+(x,y)=c_5 I_{\tilde{\omega}}(\sigma_m\rho/c)\left[\cos(\tilde{\omega}\varphi)-ctg(\tilde{\omega}\pi/2)\sin(\tilde{\omega}\varphi)\right],\ c_5\neq 0,\ m\in N, \qquad (35)$$

ko'rinishdagi formula bilan aniqlaymiz.

Bu yerdan bevosita hisoblash orqali quyidagini topamiz:

$$\begin{cases}\tau_m^+(x)=\lim\limits_{y\to+0}u_m^+(x,y)=c_5 I_{\tilde{\omega}}(\sigma_m x/c),\ x\in[0,1];\\ \nu_m^+(x)=\lim\limits_{y\to+0}\dfrac{\partial}{\partial y}u_m^+(x,y)=-c_5\tilde{\omega}ctg(\tilde{\omega}\pi/2)x^{-1}I_{\tilde{\omega}}(\sigma_m x/c),\ x\in(0,1).\end{cases} \qquad (36)$$

$u(x,y)$ funksiyaning $\bar{\Delta}$ da uzluksizligidan

$$\begin{cases}\tau_m^-(x)=\tau_m^+(x),\ x\in[0,1],\\ \nu_m^-(x)=\nu_m^+(x),\ x\in(0,1).\end{cases} \qquad (37)$$

bajariladi.

(23) va (36) larni (37) ga qo'yamiz hamda $\omega=\tilde{\omega}$ deb hisoblaymiz. Natijada c_3 va c_5 larga nisbatan bir jinsli algebraik tenglamalar sistemasiga ega bo'lamiz:

$$\begin{cases}2^{\omega}c_1+ctg\dfrac{\omega\pi}{2}c_3=0,\\ 2^{\omega}c_1-c_3=0.\end{cases} \qquad (38)$$

(38) sistemadan $ctg\frac{\omega\pi}{2}=-1$ tenglamaga ega bo‘lamiz. $\omega>0$ ekanligini inobatga olib, bu tenglamaning yechimini

$\omega_n=2n-1/2,\ n\in N$ (39)

ko‘rinishda topamiz.

(39) ni inobatga olib $\mu_n=\omega_n^2,\ n\in N$ $\{(18),\ \left|\lim\limits_{\eta\to+\infty}Y(\eta)\right|<+\infty\}$ va {(29),(30)} masalalarning xos qiymatlarini topamiz.

Ta’kidlab o‘tish kerakki, $\omega=\omega_n$ bo‘lganda (34) tenglik bilan aniqlangan $S(\varphi)$ funksiyani

$$S_n(\varphi)=\sqrt{2}\sin\left[\frac{\pi}{4}+\left(2n-\frac{1}{2}\right)\varphi\right] \qquad (40)$$

ko‘rinishda yozish mumkin.

M.S.Salohiddinov va A.Q.O‘rinovlarning [7] ishida (40) funksiyalar sistemasining $L_2(0,\pi/2)$ fazoda bazis tashkil qilinishi isbotlangan.

Yuqorida isbotlanganlarni va (22), (35), $\omega=\tilde{\omega}=\omega_n$ tengliklarni hisobga olib xulosa qilishimiz mumkinki,

$$u_{nm}(x,y)=\begin{cases} c_{5nm}\sqrt{2}\sin\left(\frac{\pi}{4}+\omega_n\varphi\right)I_{\omega_n}(\sigma_m\rho/c),\ (x,y)\in\bar{\Delta}_0, \\ c_{5nm}\left(\frac{x+y}{x-y}\right)^{\omega_n/2} I_{\omega_n}(\sigma_m\xi/c),\ (x,y)\in\bar{\Delta}_1, \end{cases} \qquad (41)$$

funksiyalar $\bar{\Delta}$ da {(9),(10)} masalaning trivial bo‘lmagan uzluksiz yechimlaridir, bu yerda $\rho=\sqrt{x^2+y^2}$, $\varphi=arctg(y/x)$, $\xi=\sqrt{x^2-y^2}$, $\omega_n=2n-\frac{1}{2}$.

U holda

$$U_{nm}(x,y,z)=u_{nm}(x,y)Z_m(z),\ n,m\in N, \qquad (42)$$

funksiyalar Ω sohada (1) tenglamaning (4)-(6) shartlarini qanoatlantiruvchi trivial bo‘lmagan yechimi hisoblanadi, bu yerda $Z_m(z)$ va $u_{nm}(x,y)$ – funksiyalar (13) va (42) tengliklar bilan aniqlanadi.

III. Masala yechimi yagonaligi

Ma’lumki, Ω sohada (1) tenglamaning (4)-(6) shartlarini qanoatlantiruvchi trivial bo‘lmagan yechimi (42) tenglik bilan aniqlanadi. Shu sababli Ω_0 sohada T masalaning $U(x,y,z)=V(\rho,\varphi,z)$ yechimini

$$V(\rho,\varphi,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}c_{3nm}I_{\omega_n}(\sigma_m\rho/c)S_n(\varphi)Z_m(z), \qquad (43)$$

qator ko‘rinishida qidiramiz, bu yerda $Z_m(z)$ va $S_n(\varphi)$ funksiyalar mos holda (13) va (40) tengliklar bilan aniqlanadi, c_{3nm} – esa hozircha noma’lum koeffitsientlar.

Faraz qilaylik (43) funksional qator $\bar{\Omega}_0$ da tekis yaqinlashuvchi bo‘lsin. U holda bu tenglikning ikkala tomonini $S_k(\varphi)z^{2\gamma}Z_l(z)$ ga ko‘paytiramiz va $\Pi=\{(\varphi,z):\varphi\in(0,\pi/2),z\in(0,c)\}$ soha bo‘yicha integrallaymiz

$$c_{3kl}I_{\omega_k}\left(\frac{\sigma_l\rho}{c}\right)\frac{1}{2}\left[cJ_{3/2-\gamma}(\sigma_l)\right]^2=\int_0^c\int_0^{\pi/2}V(\rho,\varphi,z)S_k(\varphi)z^{2\gamma}Z_l(z)d\varphi dz,\ k,l\in N. \quad (44)$$

(ρ,φ,z) silindrik koordinatalarda (3) shart quyidagi

$$V(1,\varphi,z)=f(\varphi,z),\ \varphi\in[0,\pi/2],\ z\in[0,c], \quad (45)$$

ko‘rinishda yoziladi, bu yerda $f(\varphi,z)=\tilde{F}(\cos\varphi,\sin\varphi,z)$.

(44) tenglikda $\rho=1$ qo‘yamiz va $k=n,\ \ l=m$ (bu qulaylik uchun) hamda (45) shartni inobatga olib c_{3nm} koeffitsientlarni $c_{3nm}=d_m f_{nm}/I_{\omega_n}(\sigma_m/c)$ ko’rinishda topamiz, bu yerda

$$d_m=2/\left[cJ_{3/2-\gamma}(\sigma_m)\right]^2,$$

$$f_{nm}=\int_0^c\int_0^{\pi/2}f(\varphi,z)S_n(\varphi)z^{2\gamma}Z_m(z)d\varphi dz. \quad (46)$$

Endi quyidagi teoremani isbotlashimiz mumkin.

1-teorema. Agar T masalaning yechimi mavjud bo‘lsa, u holda u yagonadir.

Isbot. Buning uchun bir jinsli T masalaning faqat trivial yechimga ega ekanligini ko‘rsatish yetarli. Aytaylik $f(\varphi,z)\equiv 0$ bo‘lsin. U holda barcha $n,m\in N$ lar uchun $c_{3nm}=0$ bo’ladi. Bu tenglikni inobatga oladigan bo‘lsak (44) dan $\int_0^c\int_0^{\pi/2}V(\rho,\varphi,z)S_n(\varphi)z^{2\gamma}Z_m(z)d\varphi dz=0$ kelib chiqadi. $Z_m(z)$ funksiyalar sistemasining $L_2(0,c)$ fazoda $z^{2\gamma}$ vazn bilan to‘laligiga asosan $\int_0^{\pi/2}V(\rho,\varphi,z)S_n(\varphi)d\varphi=0,\ n\in N$ tenglik hosil bo‘ladi. Agar (40) funksiyalar sistemasining $L_2(0,\pi/2)$ fazoda to‘laligini e’tiborga oladigan bo‘lsak, oxirgi tenglikdan $\bar{\Omega}_0$ da $V(\rho,\varphi,z)\equiv 0$ ekanligi kelib chiqadi.

$U(x,y,z)=V(\rho,\varphi,z)\equiv 0$ ekanligidan $U(x,+0,z)\equiv 0$, $x\in[0,1],\ z\in[0,c]$, $U_y(x,+0,z)\equiv 0$, $x\in(0,1),\ z\in(0,c)$ ekanligi kelib chiqadi.

U holda, $U(x,y,z)\in C(\bar{\Omega})$ ga asosan

$$U(x,-0,z)\equiv 0,\ x\in[0,1],\ z\in[0,c],\ U_y(x,-0,z)\equiv 0, x\in(0,1),\ z\in(0,c) \quad (47)$$

hosil bo‘ladi.

U holda [8] ishning natijasiga asosan

$$U_{xx} - U_{yy} + U_{zz} = 0,\ (x,y,z) \in \Omega_1,$$

tenglamaning (47) shartlarni qanoatlantiruvchi yechimi aynan nolga teng bo‘ladi, ya’ni $U(x,y,z) \equiv 0,\ (x,y,z) \in \bar{\Omega}_1$. 1-teorema isbot bo‘ldi.

IV. T masalaning yechimini qurish

$c_{3nm} = d_m f_{nm} / I_{\omega_n}(\sigma_m / c)$ ning qiymatini (41) tenglikka, so‘ngra uni (42) tenglikka qo‘yamiz va T masalaning xususiy yechimlarini hosil qilamiz:

$$U_{nm}(x,y,z) = \begin{cases} U_{nm}^{+}(x,y,z),\ (x,y,z) \in \bar{\Omega}_0,\ n,m \in N, \\ U_{nm}^{-}(x,y,z),\ (x,y,z) \in \bar{\Omega}_1,\ n,m \in N, \end{cases}$$

bu yerda,

$$U_{nm}^{+}(x,y,z) = 2S_n(\varphi) R_{nm}(\rho) \tilde{Z}_m(z) f_{nm},\ (x,y,z) \in \bar{\Omega}_0, \quad (48)$$

$$U_{nm}^{-}(x,y,z) = 2\left(\frac{x+y}{x-y}\right)^{\omega_n/2} R_{nm}(\xi) \tilde{Z}_m(z) f_{nm},\ (x,y,z) \in \bar{\Omega}_1, \quad (49)$$

$$R_{nm}(\rho) = I_{\omega_n}(\sigma_m \rho / c) / I_{\omega_n}(\sigma_m / c),\ \rho = \sqrt{x^2 + y^2},\ \xi = \sqrt{x^2 - y^2}, \quad (50)$$

$$\tilde{Z}_m(z) = z^{1/2-\gamma} J_{1/2-\gamma}(\sigma_m z / c) / \left[c J_{3/2-\gamma}(\sigma_m) \right]^2, \quad (51)$$

$S_n(\varphi)$ va f_{nm} lar esa mos holda (40) va (46) formulalar bilan aniqlangan.

2-teorema. Agar $f(\varphi,z)$ funksiya guyidagi

I. $f(\varphi,z) \in C_{\varphi,z}^{4,5}(\bar{\Pi})$, bu yerda $\Pi = \{(\varphi,z) : \varphi \in (0,\pi/2), z \in (0,c)\}$;

II. $\left.\frac{\partial^j}{\partial \varphi^j} f(\varphi,z)\right|_{\varphi=0} = 0,\ \left.\frac{\partial^j}{\partial \varphi^j} f(\varphi,z)\right|_{\varphi=\pi/2} = 0,\ j = \overline{0,3};$

III. $\left.\frac{\partial^j}{\partial z^j} f(\varphi,z)\right|_{z=0} = 0,\ \left.\frac{\partial^j}{\partial z^j} f(\varphi,z)\right|_{z=c} = 0,\ j = \overline{0,4},$

shartlarni qanoatlantirsa, u holda T masalaning yechimi mavjud va u

$$U(x,y,z) = \begin{cases} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} U_{nm}^{+}(x,y,z),\ (x,y,z) \in \bar{\Omega}_0, \\ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} U_{nm}^{-}(x,y,z),\ (x,y,z) \in \bar{\Omega}_1, \end{cases} \quad (53)$$

ko‘rinishida aniqlanadi, bu yerda $U_{nm}^{+}(x,y,z)$, $U_{nm}^{-}(x,y,z)$ – funksiyalar (48) va (49) formulalar orqali berilgan.

2-teoremaning isboti xuddi [9] ishdagi kabi bajariladi.

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