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Boundary Value Problem for a Model Parabolic-Hyperbolic Equation With Bitsadze-Samarskii Condition

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Annotation: In the present paper, we study the existence and uniqueness of a boundary value problem with Bitsadze-Samarskii condition for a model parabolic-hyperbolic equation in a bounded domain of the plane. To do this we employ the theory of integral equations.

Key words: parabolic-hyperbolic equation, boundary value problem, Bitsadze-Samarskii condition.

Introduction. The study of boundary value problems for partial differential equations (PDEs) of mixed parabolic-hyperbolic type occupies a significant place in modern mathematical physics, owing to its profound applications in modeling phenomena where diffusive and wave-like behaviors coexist. Such equations arise naturally in diverse fields, including plasma physics, transonic fluid dynamics, and the theory of composite materials, where abrupt transitions between different physical regimes necessitate a hybrid mathematical framework. Among these, equations exhibiting a composite structure–parabolic in one region and hyperbolic in another–pose unique analytical challenges, particularly when coupled with non-classical boundary conditions that reflect the interplay between distinct dynamical behaviors across interfaces.

Literature review. In [1-4] works which solvability questions of various nonlocal problems were studied for parabolic-hyperbolic type equations. Also, in [5-7] numerical aspects of local and nonlocal boundary value problems for parabolic-hyperbolic equations were studied.

K. B. Sabitov and R. M. Safina [8] studied the first boundary-value problem in a rectangle for an equation of mixed type with a singular coefficient. They established a criterion for the uniqueness of solutions and construct the solution as the sum of a series in the system of eigenfunctions of a one-dimensional eigenvalue problem.

The Bitsadze-Samarsky boundary condition, a non-local constraint introduced by Soviet mathematicians A.V. Bitsadze and A.A. Samarskii in the mid-20th century, has emerged as a cornerstone in the analysis of such problems. Unlike traditional Dirichlet or Neumann conditions, this non-local formulation connects boundary values across different regions of the domain or incorporates integral relationships, thereby encapsulating complex interactions inherent in multi-physics systems. While extensively studied for elliptic and purely hyperbolic equations, its application to parabolic-hyperbolic equations remains a topic of active research, particularly in cases where the interface between parabolic and hyperbolic regions is dynamic or singular.

In [9] work Bitsadze-Samarsky nonlocal problem for second order mixed type equation was investigated. The solution of the problem was represented by Fourier-Bessel series.

In the present paper, we aim to study unique solvability of a non-local problem with Bitsadze-Samarskii condition for a model parabolic-hyperbolic equation.

3. Main results. Let us denote by Ω the domain of the plane xOt bounded by the straight lines $x+t=0$, $x-t=l$, $x=0$, $x=l$, $t=T$, where $l=const>0$, $T=const>0$. Moreover, we introduce the following notations:

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$\Omega_1 = [\Omega \cap (t>0)] \cup AE$,

$\Omega_2 = \Omega \cap (t<0)$, $AE = \{(x,T): 0<x<l\}$,

$OA = \{(0,t): 0<t<T\}$, $OB = \{(x,0): 0<x<l\}$,

$BE = \{(l,t): 0<t<T\}$, $OM = \{(x,t): t=-x, 0<x<l/2\}$,

$BM = \{(x,t): t=x-l, l/2<x<l\}$.

In the domain Ω, we consider the following equation

(1) $u_{xx} - \frac{1}{2}(1-\operatorname{sgn} t)u_{tt} - \frac{1}{2}(1+\operatorname{sgn} t)u_t = 0$.

Since $\operatorname{sgn} t = 1$ in the domain Ω_1, the equation (1) takes the following form in the domain Ω_1

(2) $u_{xx} - u_t = 0$.

And in the domain Ω_2, it has the form

(3) $u_{xx} - u_{tt} = 0$.

(2) is the Fourier's equation and it belongs to the parabolic type. Equation (3) represents the string vibration equation, and it belongs to the hyperbolic type. So, the equation (1) belongs to mixed type in the domain Ω. Moreover, the line OB is the line of type changing of the equation (1) and it also one of the characteristics of the equation (1).

In the domain Ω, for the equation (1), we study the following non-local problem:

Problem BS. *Find a function* $u(x,t) \in C(\bar{\Omega}) \cap C^{2,1}_{x,t}(\Omega_1) \cap C^2(\Omega_2)$ *such that it satisfies equation (1) in the domains* Ω_1 *and* Ω_2*, on the line* OB *it satisfies*

(4) $\lim_{t\to+0} u_t(x,t) = \lim_{t\to-0} u_t(x,t)$, $\quad 0<x<l$,

gluing condition and the following conditions

(5) $u(0,t) = \varphi_1(t)$, $\quad 0 \le t \le T$,

(6) $u(l,t) = a(t)u(\xi_0,t) + \varphi_2(t)$, $\quad 0 \le t \le T$,

(7) $\frac{d}{dx}u\left(\frac{x}{2}, -\frac{x}{2}\right) + \alpha(x)u_t(x,0) = \beta(x)$, $\quad 0<x<l$,

where $\varphi_1(t)$, $\varphi_2(t)$, $a(t)$, $\alpha(x)$ *and* $\beta(x)$ *are given sufficiently smooth functions.*

The condition (6) is the condition of the Bitsadze-Samarsky type that connects the boundary value $u(l,t)$ of the unknown function $u(x,t)$ with the inner value $u(\xi_0,t)$ of it. Particularly, if $a(t) \equiv 0$, $t \in [0,T]$ then the first boundary value problem follows from Problem BS. So, we assume that $a(t) \neq 0$, $t \in [0,T]$.

We will study the unique solvability of the considered problem. Assume that there exists a solution $u(x,t)$ to Probem BS. Under this assumption, we will introduce the following notations

(8) $u(x,0) = \tau(x),\ 0 \le x \le l;\quad u_t(x,0) = \nu(x),\ 0 < x < l$

and the following assumptions

(9) $\tau(x) \in C[0,l] \cap C^2(0,l),\qquad \nu(x) \in C(0,l) \cap L[0,l]$.

If we assume that $\tau(x)$ and $\nu(x)$ are temporarily known functions, then $u(x,t)$ can be sought in the domain Ω_2 as a solution of the Cauchy problem for equation (3) satisfying conditions (8), in the form

(10) $u(x,t) = \frac{1}{2}\left[\tau(x+t) + \tau(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \nu(\xi)d\xi$.

For obeying (10) into the condition (7), we will compute

$$u\left(\frac{x}{2}, -\frac{x}{2}\right) = \frac{1}{2}\left[\tau(0) + \tau(x)\right] + \frac{1}{2}\int_0^x \nu(\xi)d\xi,$$

(11) $\frac{d}{dx}u\left(\frac{x}{2}, -\frac{x}{2}\right) = \frac{1}{2}\left[\tau'(x) - \nu(x)\right]$.

Substituting (11) into (7), and considering $u_t(x,0) = \nu(x),\ 0 < x < l$, we obtain the following first functional relation between unknown functions $\tau(x)$ and $\nu(x)$ that was brought from the domain Ω_2:

(12) $\tau'(x) + \left[2\alpha(x) - 1\right]\nu(x) = 2\beta(x),\quad 0 < x < l$.

Now, considering the problem in the domain Ω_1, we will pass to the limit as $t \to +0$ in equation (2) and conditions (5) and (6). As a result, we have

(13) $\begin{cases} \tau'(x) - \nu(x) = 0, 0 < x < l; \\ \tau(0) = \varphi_1(0), \tau(l) = a(0)\tau(\xi_0) + \varphi_2(0). \end{cases}$

Thus, Problem BS is equivalently reduced to the problem {(12), (13)} of unique identifying unknown functions $\tau(x)$ and $\nu(x)$ with demanded properties. If we can uniquely find unknown functions $\tau(x)$ and $\nu(x)$ from the system {(12), (13)}, then the solution to the problem BS can be represented by formula (10) in the domain Ω_2, and in the domain Ω_1 it is represented as a solution of the first-boundary value problem for the equation (2). So, from now, we will study the unique solvability of the system {(12), (13)}.

We will consider two cases:

Case 1. $\alpha(x) \equiv \frac{1}{2},\ x \in (0,l)$.

Case 2. $\alpha(x) \ne \frac{1}{2},\ x \in (0,l)$.

In the case 1, based on (12), we have

$$(14)\ \tau'(x) = 2\beta(x),\ \ 0 < x < l.$$

Integrating (14), we find $\tau(x)$ in the following form

$$(15)\ \tau(x) = 2\int_0^x \beta(\xi)d\xi + C_1,$$

where C_1 is an arbitrary constant.

By obeying (15) to the second condition of (13), we have

$$C_1 = \varphi_1(0), \qquad \tau(x) = 2\int_0^x \beta(\xi)d\xi + \varphi_1(0),$$

$$(16)\ a(0)\left(2\int_0^{\xi_0} \beta(\xi)d\xi + \varphi_1(0)\right) + \varphi_2(0) = 2\int_0^l \beta(\xi)d\xi + \varphi_1(0).$$

Thus, in this case, for the solvability of the problem {(12), (13)} we have to demand additional condition (16).

Let us assume that the condition (16) be fulfilled. Then, the function $\tau(x)$ is defined by

$$(17)\ \tau(x) = 2\int_0^x \beta(\xi)d\xi + \varphi_1(0).$$

And the function $\nu(x)$ will be find as follows

$$(18)\ \nu(x) = 2\beta'(x).$$

Substituting obtained (17) nad (18) expressions of the unknown functions $\tau(x)$ and $\nu(x)$ into (10), we will have the representation of the solution of the problem BS in the domain Ω_2.

Now, we consider the problem in the domain Ω_1.

Let us introduce the notation $u(l,t) = \varphi(t),\ 0 \le t \le T$, where $\varphi(t)$ is an unknown function. If we temporarily assume that $\varphi(t)$ is a known function then the function $u(x,t)$ can be sought as a solution of the first boundary value problem for the equation (2) in the following form:

$$u(x,t) = \int_0^l \tau(\xi)G_1(x,t;\xi,0)d\xi +$$

$$(19)\ +\int_0^t \varphi_1(\eta)G_{1\xi}(x,t;0,\eta)d\eta - \int_0^t \varphi(\eta)G_{1\xi}(x,t;l,\eta)d\eta,$$

where $G_1(x,t;\xi,\eta)$ is the Green function of the first boundary value problem that is defined by [1]:

$$G_1(x,t;\xi,\eta) = \frac{1}{2\sqrt{\pi(t-\eta)}} \times$$

$$\sum_{n=-\infty}^{+\infty}\left\{exp\left[-\frac{(x-\xi+2nl)^2}{4(t-\eta)}\right]-exp\left[-\frac{(x+\xi+2nl)^2}{4(t-\eta)}\right]\right\},\ t>\eta .$$

Using (19), we compute $u(\xi_0,t)$:

$$u(\xi_0,t)=\int_0^l \tau(\xi)G_1(\xi_0,t;\xi,0)d\xi+$$

$$+\int_0^t \varphi_1(\eta)G_{1\xi}(\xi_0,t;0,\eta)d\eta-\int_0^t \varphi(\eta)G_{1\xi}(\xi_0,t;l,\eta)d\eta .$$

Substituting this expression of $u(\xi_0,t)$ into (6), and considering $u(l,t)=\varphi(t)$, we will obtain the following integral equation with respect to the function $\varphi(t)$ in the form

$$(20)\ \ \varphi(t)=\int_0^t \varphi(\eta)\left[a(t)G_{1\xi}(\xi_0,t;l,\eta)\right]d\eta=f(t),\qquad 0\le t\le T,$$

where

$$f(t)=\varphi_2(t)+a(t)\left[\int_0^l \tau(\xi)G_1(\xi_0,t;\xi,0)d\xi+\int_0^t \varphi_1(\eta)G_{1\xi}(\xi_0,t;0,\eta)d\eta\right].$$

(20) is the second kind Volterra integral equation and its kernel has the form

$$a(t)G_{1\xi}(\xi_0,t;l,\eta)=\frac{a(t)}{\sqrt{\pi}}\sum_{n=-\infty}^{+\infty}\left\{\frac{\xi_0-l-2nl}{4(t-\eta)^{3/2}}\times\exp\left[-\frac{(\xi_0-l-2nl)^2}{4(t-\eta)}\right]+\frac{\xi_0+l-2nl}{4(t-\eta)^{3/2}}\exp\left[-\frac{\left(\xi_0+l-2nl\right)^2}{4(t-\eta)}\right]\right\}$$

$\forall\xi_0\in(0,l)$ and $n\in Z$, we have $\xi_0\pm l-2nl\neq 0$, and for the values $t>\eta$ the series in it converges. Moreover, for all $n\in Z$, we have

$$\lim_{\eta\to t}\frac{\xi_0\pm l-2nl}{4(t-\eta)^{3/2}}\exp\left[-\frac{(\xi_0\pm l-2nl)^2}{4(t-\eta)}\right]=0.$$

Thus, the kernel $a(t)G_{1\xi}(\xi_0,t;l,\eta)$ is continuous in $\{(t,\eta):0\le\eta<t\le T\}$ and bounded, and also

$$\lim_{\eta\to t}a(t)G_{1\xi}(\xi_0,t;l,\eta)=0.$$

Now, let us study the function $f(t)$.

Here the function $K(\xi,t)$ has the form

$$K(\xi,t)=\frac{1}{2\sqrt{\pi t}}\left\{\sum_{n=-\infty}^{+\infty}\exp\left[-\frac{(\xi_0-\xi-2nl)^2}{4t}\right]-\sum_{n=-\infty}^{+\infty}\exp\left[-\frac{(\xi_0+\xi-2nl)^2}{4t}\right]\right\}.$$

It is continuous, bounded in $\{(\xi,t):0\le\xi\le l,0<t\le T\}$, and $\lim_{t\to\infty}K(\xi,t)=0$.

Then the first integral in the expression of the function $f(t)$ can be written in the form

$$I_1(t)=\int_0^l \tau(\xi)G_1(\xi_0,t;\xi,0)d\xi=$$

$$=\frac{1}{2\sqrt{\pi t}}\int_0^l \tau(\xi)\exp\left[-\frac{(\xi_0-\xi)^2}{4t}\right]d\xi+\int_0^l \tau(\xi)K(\xi,t)d\xi .$$

Changing the variable of integration by formula $\xi-\xi_0=2s\sqrt{t}$, we obtain:

$$I_1(t)=\frac{1}{\sqrt{\pi t}}\int_{-\xi_0/2\sqrt{t}}^{(l-\xi_0)/2\sqrt{t}} \tau(\xi_0+2\sqrt{t}s)e^{-s^2}ds+\int_0^l \tau(\xi)K(\xi,t)d\xi .$$

Based on the properties of the functions $\tau(\xi)$ and $K(\xi,t)$, it follows that $I_1(t)\in C[0,T]$.

Since

$$G_1(\xi_0,t;0,\eta)\in C(0\le\eta<t\le T),\ \lim_{\eta\to t}G_{1\xi}(\xi_0,t;0,\eta)=0 \text{ and } \varphi_1(t)\in C[0,T]$$

then the second integral of the expression of the function $f(t)$ is also continuous. Considering above facts and the continuity of the functions $a(t)$, $\varphi_2(t)$, we can conclude that $f(t)\in C[0,T]$.

Thus, (20) is a second kind Volterra integral equation with continuous right-hand side and kernel. Then, based on the theory of integral equations, this integral equation has unique, continuous solution.

Substituting the obtained expression of $\varphi(t)$ from the integral equation (20) into (19), we have the representation of the solution of problem BS in the domain Ω_1.

Thus, for the case 1, we have proved the following theorem:

Theorem 1. *Let* $\varphi_1(t)\in C[0,T]$, $\varphi_2(t)\in C[0,T]$, $a(t)\in C[0,T]$, $\beta(x)\in C[0,l]\cap C^1(0,l)$, $\alpha\equiv\frac{1}{2}$ *and the equality (16) be valid.*

Then, the solution to the problem BS exists and is unique.

Case 2. Let $\alpha\neq\frac{1}{2}$, $x\in(0,l)$.

In this case, from (12) and (13), we will have the following problem with respect to the unknown function $\tau(x)$:

(21) $\tau''(x)+\alpha_1(x)\tau'(x)=\beta_1(x),\ \ 0<x<l$,

(22) $\tau(0)=\varphi_1(0),\quad \tau(l)=a(0)\tau(\xi_0)+\varphi_2(0)$,

where

$$\alpha_1(x)=\frac{1}{2\alpha(x)-1},\qquad \beta_1(x)=\frac{2\beta(x)}{2\alpha(x)-1}.$$

Let us introduce notation $\tau'(x)=z(x)$. Then, from (21), we have

(23) $z'(x)+\alpha_1(x)z(x)=\beta_1(x),\quad 0<x<l$.

It is straightforward to see that the general solution of the equation (23) has the following form

$$z(x) = C_1 e^{-\int_0^x \alpha_1(t)dt}.$$

From the last, considering the equality $\tau'(x) = z(x)$, we will find the function $\tau(x)$ in the following form:

$$(24)\ \tau(x) = C_1 \int_0^x e^{-\int_0^t \alpha_1(\eta)d\eta} dt + C_2,$$

where C_1 and C_2 are arbitrary constants. To define the constants C_1 and C_2, we obey the solution (24) into the condition (22):

From the condition $\tau(0) = \varphi_1(0)$, it follows that $C_2 = \varphi_1(0)$. Using (24), we compute

$$\tau(l) = C_1 \int_0^l e^{-\int_0^t \alpha_1(\eta)d\eta} dt + \varphi_1(0),$$

$$\tau(\xi_0) = C_1 \int_0^{\xi_0} e^{-\int_0^t \alpha_1(\eta)d\eta} dt + \varphi_1(0).$$

Substituting the expressions of $\tau(l)$ and $\tau(\xi_0)$ into the second condition of (22), we have

$$C_1 \int_0^l e^{-\int_0^t \alpha_1(\eta)d\eta} dt + \varphi_1(0) = a(0) C_1 \int_0^{\xi_0} e^{-\int_0^t \alpha_1(\eta)d\eta} dt +$$

$$(25)\ + a(0)\varphi_1(0) + \varphi_2(0).$$

If the following

$$(26)\ \int_0^l e^{-\int_0^t \alpha_1(\eta)d\eta} dt - a(0) \int_0^{\xi_0} e^{-\int_0^t \alpha_1(\eta)d\eta} dt \neq 0$$

relation holds, then the unknown C_1 is uniquely defined from the equality (25). Substituting obtained expressions of C_1 and C_2 into (24), we uniquely define unknown function $\tau(x)$. Then, the function $\nu(x)$ will be determined according to (12), in the following form:

$$(26)\ \nu(x) = \beta_1(x) - \varphi_1(0)\alpha_1(x) e^{-\int_0^x \alpha_1(t)dt}.$$

Then the study of the problem BS can be continued in a similar way in the case 1.

Conclusion. Our approach shows that the unique solvability of the non-local problem is connected with the properties of the functions involved in the boundary conditions.

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