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# Parabolo-Giperbolik Tenglama Uchun Umumlashgan Bitsadze-Samarskiy Masalasi

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**Annotatsiya:** Ushbu maqolada aralash parabolo-giperbolik tenglama uchun umumlashgan Bitsadze-Samarskiy masalasi qaralgan. Masalaning bir qiymatli yechilishi integral tenglamalar nazariyasi yordamida isbotlangan.

**Kalit so'zlar:** aralash turdagi tenglama, parabolo-giperbolik tenglama, Bitsadze–Samarskiy masalasi, integral tenglamalar nazariyasi, bir qiymatli yechim, giperbolik soha, parabolik soha, ulash sharti, chegaraviy shart, Koshi masalasi.

**Kirish. Adabiyotlar tahlili.** Plazma nazariyasida yuzaga keladigan elliptik tenglamalar uchun yangi noan'anaviy chegaraviy masala [1] da shakllantirilgan bo'lib, hozirda u Bitsadze-Samarskiy masalasi nomi bilan mashhur. Ushbu turdagi masalalar hamda ularning turli umumlashtirilgan ko'rinishlari ko'plab mualliflar tomonidan o'rganilgan. Batafsil ma'lumot uchun [2], [3] ishlar va ulardagi adabiyotlarni tavsiya etamiz. Bitsadze-Samarskiy masalalariga bag'ishlangan tadqiqotlar soni soni anchagina bo'lishiga qaramay, aralash turdagi tenglamalar uchun hali yetarlicha o'rganilmagan.

Bunday masalalarning murakkabligi, odatda, ularni yechish Fredgolm yoki Volterra integral tenglamalariga**,** ba'zida esa bir noma'lumli singular integral tenglamalarga olib kelishi bilan bog'liq. Mazkur integral tenglamalar uchun yechilish nazariyasi yetarlicha ishlab chiqilgan bo'lsa-da, ularni aralash turdagi muhitda qo'llash muayyan texnik va nazariy qiyinchiliklarni keltirib chiqaradi.

Mazkur maqolada aralash tipga tegishli model parabola-giperbolik tenglama uchun noma'lum funksiyaning soha chegarasidagi qiymatini sohaning bir nechta ichki nuqtalaridagi qiymatlari bilan bog'lovchi Bitsadze-Samarskiy shartli masala bayon qilinadi va tadqiq etiladi.

**Masalaning qo'yilishi.** $\Omega$ orqali $xOt$ tekisligining $x+t=0$, $x-t=l$, $x=0$, $x=l$, $t=T$ chiziqlar bilan chegaralangan sohasini belgilaylik, bu yerda $l=const>0$, $T=const>0$. Shuningdek, quyidagi belgilashlarni kiritaylik:

$$\Omega_1=\left[\Omega\cap(t>0)\right]\cup AE,$$

$$\Omega_2=\Omega\cap(t<0),\quad AE=\{(x,T):0<x<l\},$$

$$OA=\{(0,t):0<t<T\},\quad OB=\{(x,0):0<x<l\},$$

$$BE=\{(l,t):0<t<T\},\quad OM=\{(x,t):t=-x,0<x<l/2\},$$

$$BM=\{(x,t):t=x-l,l/2<x<l\}.$$

$\Omega$ sohada quyidagi

$$u_{xx}-\frac{1}{2}(1-\operatorname{sgn}t)u_{tt}-\frac{1}{2}(1+\operatorname{sgn}t)u_t=0 \tag{1}$$

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tenglamani qaraymi. $\Omega_1$ sohada $\operatorname{sgn} t = 1$ ekanligini e'tiborga olsak, (1) tenglama

$$u_{xx} - u_t = 0 \qquad (2)$$

ko'rinishni oladi.

$\Omega_2$ sohada esa $\operatorname{sgn} t = -1$ ekanligidan (1) tenglama

$$u_{xx} - u_{tt} = 0 \qquad (3)$$

ko'rinishda bo'ladi.

(2) tenglama Furye tenglamasi bo'lib, u parabolik tipga tegishlidir. (3) tenglama esa tor tebranish tenglamasi bo'lib, u giperbolik tipga tegishlidir. Shuning uchun (1) tenglama $\Omega$ sohada aralash tenglama, ya'ni parabolo - giperbolik tenglama bo'lib, $OB$ kesma uning tip o'zgarish chizig'idir. (1) tenglamaning xarakteristakalari $\Omega_1$ va $\Omega_2$ sohalarda mos ravishda $t = const$ va $x \pm t = const$ chiziqlardan iborat bo'lib, tenglamaning tip o'zgarish chizig'i, ya'ni $OB$ kesma xarakteristika ham bo'ladi.

(1) tenglama uchun $\Omega$ sohada quyidagi masalani tadqiq qilamiz:

**Umumlashgan BS masala.** *Shunday* $u(x,t) \in C(\bar{\Omega}) \cap C^{2,1}_{x,t}(\Omega_1) \cap C^{2,2}_{x,t}(\Omega_2)$ *funksiya topilsinki, u* $\Omega_1$ *va* $\Omega_2$ *sohalarda (1) tenglamani,* $OB$ *tip o'zgarish chizig'ida*

$$\lim_{t\to+0} u_t(x,t) = \lim_{t\to-0} u_t(x,t), \qquad 0 < x < l, \qquad (4)$$

*ulash shartlarini hamda*

$$u(0,t) = \varphi_1(t), \qquad 0 \le t \le T, \qquad (5)$$

$$u(l,t) = \sum_{k=1}^{n} a_k(t) u(\xi_k, t) + \varphi_2(t), \quad 0 \le t \le T, \qquad (6)$$

$$u(x,t)\big|_{\overline{OM}} = \psi(x), \qquad 0 \le x \le \frac{l}{2} \qquad (7)$$

*shartlarni qanoatlantirsin, bu yerda* $\varphi_1(t)$, $\varphi_2(t)$, $a_k(t)$, *va* $\psi(x)$ *lar berilgan yetarlicha silliq funksiyalar bo'lib,* $\varphi_1(0) = \psi(0)$ *kelishuv sharti bajariladi.*

(6) – Bitsadze–Samarskiy sharti bo'lib, u $u(x,t)$ noma'lum funksiyaning $u(l,t)$ chegaraviy qiymatini soha ichidagi $u(\xi_k, t)$, $k = 1, 2, \ldots, n$ qiymatlari bilan bog'laydi. Agar $a_k(t) \equiv 0$, $k = \overline{1,n}$, $t \in [0,T]$ bo'lsa, birinchi chegaraviy masala kelib chiqadi. Shuning uchun $a(t) \ne 0$, $k = \overline{1,n}$, $t \in [0,T]$ deb hisoblaymiz.

Qo'yilgan masalaning bir qiymatli yechilishini tekshiramiz. Faraz qilaylik masalaning $u(x,t)$ yechimi mavjud bo'lsin. U holda, masala shartlariga asoslanib, quyidagi belgilsh va farazlarni qabul qilamiz:

$$u(x,0) = \tau(x), \; 0 \le x \le l; \qquad u_t(x,0) = \nu(x), \; 0 < x < l, \qquad (8)$$

$\tau(x)\in C[0,l]\cap C^2(0,l)$, $\nu(x)\in C(0,l)\cap L[0,l]$. (9)

U holda $u(x,t)$ funksiyani $\Omega_2$ sohada (3) tenglama uchun (8) shartlar bilan qo‘yilgan Koshi masalasining yechimi sifatida [4] quyidagi ko‘rinishda qidirishimiz mumkin:

$$u(x,t)=\frac{1}{2}\left[\tau(x+t)+\tau(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t}\nu(\xi)d\xi \quad (10)$$

(10) tenglik bilan aniqlangan $u(x,t)$ funksiyani (7) chegaraviy shartga bo‘ysundirsak, quyidagiga ega bo‘lamiz:

$$u|_{\overline{OM}}=u(x,-x)=\frac{1}{2}\left[\tau(0)+\tau(2x)\right]+\frac{1}{2}\int_{2x}^{0}\nu(\xi)d\xi=\psi(x),\quad 0\le x\le\frac{l}{2},$$

ya’ni ushbu

$$\tau(0)+\tau(2x)-\int_{0}^{2x}\nu(\xi)d\xi=2\psi(x),\quad 0\le x\le\frac{l}{2}$$

tenglik kelib chiqadi. Bu tenglikda $2x=z\in[0,l]$ belgilashni kiritib, so‘ngra $z$ o‘zgaruvchi bo‘yicha hosila olish amalini bajarsak, $\tau(x)$ va $\nu(x)$ noma’lum funksiyalar orasidagi $\Omega_2$ sohadan olingan quyidagi asosiy funksional munosabatga ega bo‘lamiz:

$$\nu(z)=\tau'(z)-\psi'(z/2),\quad 0<z<l. \quad (11)$$

Endi $u(x,t)\in C(\overline{\Omega})$ va $u_t(x,t)\in C(\Omega)$ ekanligini e’tiborga olib, (2) tenglamada va (5), (6) shartlarda $t$ ni nolga intiltiramiz. Bunda (8) belgilashlarni va (4) ulash shartini e’tiborga olsak, quyidagi

$$\tau''(z)-\nu(z)=0,\quad 0<z<l; \quad (12)$$

$$\tau(0)=\varphi_1(0),\quad \tau(l)=\sum_{k=1}^{n}a_k(0)\tau(\xi_k)+\varphi_2(0) \quad (13)$$

munosabatlar kelib chiqadi.

Agar (11), (12) va (13) tengliklardan foydalanib, $\tau(x)\in C[0,l]\cap C^2(0,l)$ va $\nu(x)\in C^1(0,l)\cap L[0,l]$ noma’lum funksiyalar bir qiymatli topilsa, u holda masalaning yechimi giperbolik sohada (10) formula bilan, parabolic sohada esa birinchi chegaraviy masalaning yechimi sifatida aniqlanadi. Shu sababli bundan buyon (11), (12) va (13) munosabatlardan $\tau(x)\in C[0,l]\cap C^2(0,l)$ va $\nu(x)\in C^1(0,l)\cap L[0,l]$ noma’lum funksiyalarni bir qiymatli aniqlash masalasi bilan shug‘ullanamiz.

Shu maqsadda, $\nu(z)$ ning (11) ifodasini (12) tenglikka qo‘ysak, $\tau(x)$ noma’lum funksiyaga nisbatan

$$\tau''(z)-\tau'(z)=-\psi'(z/2),\quad z\in(0,l) \quad (14)$$

oddiy differensial tenglama hosil bo‘ladi. Bu tenglama yechimini o‘zgarmasni variatsiyalash usulidan foydalanib topamiz. Shu maqsadda, (14) tenglamaning yechimini

$$\tau(z) = C_1(z) + C_2(z)e^z \qquad (15)$$

ko‘rinishda qidiramiz. (15) funksiyani (14) tenglamaga qo‘yib, $C_1'(z)$ va $C_2'(z)$ noma’lum funksiyalarga nisbatan quyidagi

$$\begin{cases} C_1'(z) + C_2'(z)e^x = 0 \\ C_2'(z) = -\psi'(z/2)e^{-x} \end{cases}$$

tenglamalar sistemasiga ega bo‘lamiz. Bu sistemani yechib, $C_1(z)$ va $C_2(z)$ noma’lum funksiyalarni quyidagi ko‘rinishda topamiz:

$$C_1(z) = 2\psi(z/2) + C_3, \quad C_2(z) = -\int_0^t \psi'(t/2)e^{-t}dt + C_4,$$

bu yerda $C_3$ va $C_4$ – ixtiyoriy haqiqiy sonlar. Yuqorida topilgan ifodalarni (15) tenglikga qo‘yib, bir jinsli bo‘lmagan (14) tenglamaning quyidagi umumiy yechimiga ega bo‘lamiz:

$$\tau(z) = C_3 + C_4e^z + 2\psi(z/2) - \int_0^z \psi'(t/2)e^{z-t}dt. \qquad (16)$$

Endi (16) funksiyani (13) shartlarga bo‘ysindirib, $C_3$ va $C_4$ larga nisbatan ushbu

$$\begin{cases} C_3 + C_4 = -\psi(0) \\ \left[1 - \sum_{k=1}^{n} a_k(0)\right]C_3 + \left[e^l - \sum_{k=1}^{n} a_k(0)e^{\xi_k}\right]C_4 = D \end{cases} \qquad (17)$$

tenglamalar sistemasiga ega bo‘lamiz, bu yerda

$$D = \varphi_2(0) + \sum_{k=1}^{n} a_k(0)\left[2\psi\left(\frac{\xi_k}{2}\right) - \int_0^{\xi_k} \psi'\left(\frac{t}{2}\right)e^{\xi_k - t}dt\right].$$

Agar

$$1 - \sum_{k=1}^{n} a_k(0) \neq e^l - \sum_{k=1}^{n} a_k(0)e^{\xi_k}$$

munosabat o‘rinli bo‘lsa, (17) sistemaning asosiy determinanti noldan farqli bo‘lib, bu sistemadan $C_3$ va $C_4$ noma’lumlar bir qiymatli topiladi. Ularning topilgan qiymatini (16) tenglikka qo‘yib, {(12), (13)} masala yechimi bir qiymatli aniqlanadi.

$\tau(z)$ ning topilgan ifodasini (11) tenglikka qo‘yib, $\nu(z)$ funksiyani aniqlaymiz. Shundan so‘ng masalaning yechimi $\Omega_2$ sohada (10) formula bilan aniqlanadi.

Endi $\Omega_1$ sohada yechimni aniqlashga o‘taylik.

$u(l,t) = \varphi(t)$, $0 \le t \le T$ belgilash kiritaylik, bu yerda $\varphi(t)$ noma’lum funksiya. Agar $\varphi(t)$ funksiyani vaqtincha ma’lum funksiya deb hisoblasak, u holda $u(x,t)$ funksiyani (2) tenglama uchun birinchi chegaraviy masalaning yagona yechimi sifatida quyidagicha yozishimiz mumkin [5]:

$$u(x,t)=\int_0^l \tau(\xi)G_1(x,t;\xi,0)d\xi+$$

$$+\int_0^t \varphi_1(\eta)G_{1\xi}(x,t;0,\eta)d\eta-\int_0^t \varphi(\eta)G_{1\xi}(x,t;l,\eta)d\eta \quad (18)$$

bu yerda $G_1(x,t;\xi,\eta)$ - birinchi chegaraviy masalaning Grin funksiyasi bo'lib, quyidagi ko'rinishda aniqlanadi [5]:

$$G_1(x,t;\xi,\eta)=\frac{1}{2\sqrt{\pi(t-\eta)}}\times$$

$$\sum_{n=-\infty}^{+\infty}\left\{exp\left[-\frac{(x-\xi+2nl)^2}{4(t-\eta)}\right]-exp\left[-\frac{(x+\xi+2nl)^2}{4(t-\eta)}\right]\right\},\ t>\eta.$$

(18) formula bo'yicha $u(\xi_k,t)$ larni topamiz:

$$u(\xi_k,t)=\int_0^l \tau(\xi)G_1(\xi_k,t;\xi,0)d\xi+$$

$$+\int_0^t \varphi_1(\eta)G_{1\xi}(\xi_k,t;0,\eta)d\eta-\int_0^t \varphi(\eta)G_{1\xi}(\xi_k,t;l,\eta)d\eta.$$

$u(\xi_k,t)$ ning bu ifodasini (6) shartga qo'yib, $u(l,t)=\varphi(t)$ belgilashni e'tiborga olib, $\varphi(t)$ noma'lum funksiyaga nisbatan

$$\varphi(t)=\sum_{k=1}^{n}a_k(t)\int_0^t \varphi(\eta)G_{1\xi}(\xi_k,t;l,\eta)d\eta=f(t),\ 0\le t\le T \quad (19)$$

ko'rinishdagi integral tenglamaga ega bo'lamiz, bu yerda

$$f(t)=\varphi_2(t)+\sum_{k=1}^{n}a_k(t)\left[\int_0^l \tau(\xi)G_1(\xi_k,t;\xi,0)d\xi+\int_0^t \varphi_1(\eta)G_{1\xi}(\xi_k,t;0,\eta)d\eta\right].$$

(19)- ikkinchi turdagi Volterra integral tenglamasi bo'lib, uning yadrosi quyidagi ko'rinishga ega:

$$\sum_{k=1}^{n}a_k(t)G_{1\xi}(\xi_k,t;l,\eta)=\frac{1}{\sqrt{\pi}}\sum_{k=1}^{n}a_k(t)\times$$

$$\times\sum_{n=-\infty}^{+\infty}\left\{\frac{\xi_k-l-2nl}{4(t-\eta)^{3/2}}\times\exp\left[-\frac{(\xi_k-l-2nl)^2}{4(t-\eta)}\right]+\frac{\xi_k+l-2nl}{4(t-\eta)^{3/2}}\exp\left[-\frac{(\xi_k+l-2nl)^2}{4(t-\eta)}\right]\right\}.$$

$\forall\xi_k\in(0,l)$ va $n\in Z$ uchun $\xi_k\pm l-2nl\neq 0$ bo'lib, $t>\eta$ bo'lganda undagi qator tekis yaqinlashadi. Shuningdek, $n\in Z$ uchun

$$\lim_{\eta\to t}\frac{\xi_k \pm l-2nl}{4(t-\eta)^{3/2}}\exp\left[-\frac{(\xi_k \pm l-2nl)^2}{4(t-\eta)}\right]=0.$$

Demak, $\sum_{k=1}^{n} a_k(t)G_{1\xi}(\xi_k,t;l,\eta)$ yadro $\{(t,\eta): 0\le\eta<t\le T\}$ sohada uzluksiz va chegaralangan bo'lib,

$$\lim_{\eta\to t}\sum_{k=1}^{n} a_k(t)G_{1\xi}(\xi_k,t;l,\eta)=0.$$

Endi $f(t)$ funksiyani o'rganaylik. Grin funksiyaning ko'rinishiga asosan

bu yerda $K(\xi,t)$ funksiya

$$K(\xi,t)=\frac{1}{2\sqrt{\pi t}}\left\{\sum_{n=-\infty}^{+\infty}\exp\left[-\frac{(\xi_k-\xi-2nl)^2}{4t}\right]-\sum_{n=-\infty}^{+\infty}\exp\left[-\frac{(\xi_k+\xi-2nl)^2}{4t}\right]\right\}$$

ko'rinishga ega bo'lib, u $\{(\xi,t): 0\le\xi\le l, 0<t\le T\}$ sohada uzluksiz, chegaralangan va $\lim_{t\to\infty}K(\xi,t)=0$.

U holda $f(t)$ funksiya tarkibidagi birinchi integralni

$$I_1(t)=\sum_{k=1}^{n} a_k(t)\int_0^l \tau(\xi)G_1(\xi_k,t;\xi,0)d\xi=$$

$$=\frac{1}{2\sqrt{\pi t}}\sum_{k=1}^{n} a_k(t)\left[\int_0^l \tau(\xi)\exp\left[-\frac{(\xi_k-\xi)^2}{4t}\right]d\xi+\int_0^l \tau(\xi)K(\xi,t)d\xi\right]$$

ko'rinishda yozish mumkin. Bu yerdagi 1-integralda $\xi-\xi_k=2s\sqrt{t}$ almashtirish bajaramiz:

$$I_1(t)=\frac{1}{\sqrt{\pi t}}\sum_{k=1}^{n} a_k(t)\left[\int_{-\xi_k/2\sqrt{t}}^{(l-\xi_k)/2\sqrt{t}} \tau(\xi_k+2\sqrt{t}s)e^{-s^2}ds+\int_0^l \tau(\xi)K(\xi,t)d\xi\right].$$

$\tau(\xi)$ va $K(\xi,t)$ funksiyalarning xossalariga asosan oxirgi tenglikdan $I_1(t)\in C[0,T]$ degan xulosaga kelamiz.

$G_1(\xi_k,t;0,\eta)\in C(0\le\eta<t\le T)$, $\lim_{\eta\to t}G_{1\xi}(\xi_k,t;0,\eta)=0$ va $\varphi_1(t)\in C[0,T]$ bo'lganligi uchun $f(t)$ funksiya tarkibidagi ikkinchi integral ham uzluksiz funksiyadir. Yuqoridagilarni va $a_k(t)$, $\varphi_2(t)$ funksiyalarning uzluksizligini hisobga olsak, $f(t)\in C[0,T]$ ekanligi kelib chiqadi.

Demak, (19)-yadrosi va o'ng tomoni uzluksiz bo'lgan ikkinchi tur Volterra integral tenglamasidir [6]. Integral tenglamalar nazaryasiga ko'ra bu tenglama yagona uzluksiz yechimga ega bo'ladi.

(19) integral tenglamada topilgan $\varphi(t)$ ni (18) formulaga qo'ysak, umumlashgan BS masalaning $\Omega_1$ sohadagi yechimiga ega bo'lamiz.

Shunday qilib, quyidagi teoremani isbotladik:

**Teorema.** *Agar* $\varphi_1(t)$, $\varphi_2(t)$, $a_k(t)\in C[0,T]$, $\psi(x)\in C[0,l/2]\cap C^2(0,l/2)$, $\psi'(x)\in L[0,l/2]$ *shartlar va*

$$1-\sum_{k=1}^{n}a_k(0)\neq e^{l}-\sum_{k=1}^{n}a_k(0)e^{\xi_k}$$

*munosabat bajarilsa, u holda umumlashgan BS masalasi yagona yechimga ega bo'ladi.*

**Foydalanilgan adabiyotlar**

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