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IKKINCHI TARTIBLI SPEKTRAL PARAMETRLI PARABOLO-GIPERBOLIK TIPDAGI TENGLAMA UCHUN INTEGRAL SHARTLI MASALA

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Annotatsiya. *Ushbu maqolada ikkinchi tartibli spektral parametrli parabola-giperbolik tenglama uchun integral shartli masala qaralgan. Tenglama uchun qo'yiladigan masalani bayon qilishda va o'rganishda operatorlar va ularning xossalaridan keng foydalanilgan. Masalaning bir qiymatli yechilishi esa integral tenglamalar nazariyasi yordamida isbotlangan.*
Kalit so'zlar. *Parabolo-giperbolik tipdagi tenglama, 1-tur Bessel funksiyasi, Bessel-Klifford funksiyasi, integral shartli masala, Volterra integral tenglamasi .*

Kirish

Jahonda so'nggi yillarda tadqiqotchilar differensial tenglamalar uchun yangi masalalarni, jumladan, nolokal shartli masalalarni bayon qilish va tadqiq etish bilan faol shug'ullanmoqdalar. Bunda integral shartli masalalarni qo'yish va o'rganish muhim ahamiyat kasb etadi. Hozirgi kunga qadar integral shartli masalalarni bayon qilish va tadqiq etish usullari hamda ularning amaliy ahamiyatiga bag'ishlangan bir qancha ilmiy maqolalar chop etilgan. Ularning ko'pchiligida, asosan, giperbolik, parabolik va elliptik tipdagi ikkinchi tartibli xususiy hosilali differensial tenglamalar, uchinchi tartibli xususiy hosilali differensial tenglamalar va integro-differensial tenglamalar qaralgan. Hozirgi vaqtda xususiy hosilali aralash tipdagi differensial tenglamalar uchun integral shartli masalalarni o'rganishga alohida e'tibor berilmoqda.

xOy tekislikning $y \geq 0$ qismida $x=0,\ y=1,\ \ x=1$ to'g'ri chiziqlar bilan, $y \leq 0$ qismida esa tor tebranish tenglamasining $x+y=0,\ x-y=1$ xarakteristikalari bilan chegaralangan sohani D deb belgilaylik.

$$0=\begin{cases} u_{xx}-u_y-\lambda_1^2 u,\ (x,y)\in D_1 \\ u_{xx}-u_{yy}+\lambda_2^2 u,\ (x,y)\in D_2 \end{cases} \tag{1}$$

bu yerda λ_1 va λ_2 - berilgan haqiqiy sonlar.

1-masala. Ushbu xossalarga ega bo‘lgan $u(x,y)$ funksiya topilsin:

1) $u(x,y)\in C(\bar{D})\cap C^{2,1}_{x,y}(D_1)\cap C^{2,2}_{x,y}(D_2)$;

2) $u(x,y)$ – $D_1\cup D_2$ da (1) tenglamaning regulyar yechimi;

3) $u(x,y)$ funksiya quydagi shartlarni qanoatlantirsin:

$$\int_0^1 u(x,y)dx=\int_0^y u(0,t)dt+\mu_1(y),\qquad 0\le y\le 1 \quad (2)$$

$$u(1,y)=\mu_2(y),\qquad 0\le y\le 1 \quad (3)$$

$$u(x,-x)=\psi(x),\qquad 0\le x\le (1/2); \quad (4)$$

$$\lim_{y\to+0}u_y(x,y)=\lim_{y\to-0}u_y(x,y),\qquad 0<x<1, \quad (5)$$

bu yerda $\mu_1(x),\ \ \mu_2(x),\ \ \psi(x)$ - berilgan silliq funksiyalarю

Masalaning bir qiymatli yechilishini tadqiq qilamiz. Faraz qilaylik, $u(x,y)$ funksiya qo‘yilgan masalaning yechimi bo‘lsin. Masala shartlariga asoslanib, quyidagi belgilash va farazlarni kiritaylik:

$$u(x,+0)=u(x,+0)=\tau(x),\ \ 0\le x\le 1,\quad \tau(x)\in C[0,1]\cap C^2(0,1);$$

$$\lim_{y\to+0}u_y(x,y)=\lim_{y\to-0}u_y(x,y)=\nu(x),\ \ 0<x<1,\quad \nu(x)\in C^1(0,1)\cap L[0,1]. \quad (6)$$

Ma’lumki, D_2 sohada $u_{xx}-u_{yy}+\lambda_2^2u=0$ tenglama uchun qo‘yilgan Koshi masalasining yechimi[1]

$$u(x,y)=\frac{1}{2}\left[\tau(x+y)+\tau(x-y)\right]+$$
$$+\frac{1}{2}\lambda_2\int_{x-y}^{x+y}\nu(t)J_0\left[\lambda_2\sqrt{(x-t)^2-y^2}\right]dt+\frac{1}{4}\lambda_2^2y\int_{x-y}^{x+y}\tau(t)\bar{J}_1\left[\lambda_2\sqrt{(x-t)^2-y^2}\right]dt, \quad (7)$$

ko‘rinishda yoziladi, bu yerda $J_0(x),\ \ J_1(x)$ - 1-tur Bessel funksiyasilari, $\bar{J}_1(x)=(2/x)J_1(x)$ - Bessel-Klifford funksiaysi.

Koshi masalasi yechimi ko‘rinishi (7) funksiyani (4) shartga bo‘ysundiramiz:

$$u(x,-x)=\frac{1}{2}\left[\tau(0)+\tau(2x)\right]-\frac{1}{4}\lambda_2^2x\int_{2x}^{0}\tau(t)\bar{J}_1\left[\lambda_2\sqrt{t(t-2x)}\right]dt+$$
$$+\frac{1}{2}\lambda_2\int_{2x}^{0}\nu(t)J_0\left[\lambda_2\sqrt{t(t-2x)}\right]dt=\psi(x),$$
$$0\le x\le (1/2).$$

Bu tenglikda $2x=z\in[0,1]$ almashtirish qilib, $\tau(0,0)=\psi(0)$ va $J_0'(x)=-J_1'(x)$ tengliklarni e’tiborga olsak, quyidagi tenglikka ega bo‘lamiz:

$$\tau(z)+\int_0^z\frac{z}{t}\tau(t)\frac{\partial}{\partial z}J_0\left[\lambda_2\sqrt{t(t-2x)}\right]dz-$$
$$-\int_0^z\nu(t)J_0\left[\lambda_2\sqrt{t(t-2x)}\right]dt=2\psi(z/2)-\psi(0),$$
$$0\le z\le 1.$$

$B_{mx}^{n,\lambda}$ operator belgisidan foydalansak, oxirgi tenglikni

$$B_{0x}^{0,\lambda_2}\left[\tau(x)\right]-\int_0^x B_{0x}^{1,\lambda_2}\left[\nu(t)\right]dt=2\psi(x/2)-\psi(0),\qquad 0\le x\le 1. \tag{8}$$

ko‘rinishda yozish mumkin. Bu yerda [3]

$$B_{mx}^{n,\lambda}\left[f(x)\right]\equiv f(x)+\int_m^x f(t)\left(\frac{x-m}{t-m}\right)^{1-n}\frac{\partial}{\partial x}J_0\left[\lambda_2\sqrt{(t-m)(t-x)}\right]dt, \tag{9}$$

(8) tenglikni x bo‘yicha differensiallaymiz va har ikkala tomoniga A_{0x}^{1,λ_2} operatorni tatbiq qilamiz [3],

$$A_{0x}^{1,\lambda_2}\left[f(x)\right]\equiv f(x)-\int_0^x f(t)\frac{t}{x}\frac{\partial}{\partial t}J_0\left[\lambda_2\sqrt{x(x-t)}\right]dt \tag{10}$$

so‘ngra

$$A_{0x}^{1,\lambda_2}\left\{B_{0x}^{1,\lambda_2}\left[f(x)\right]\right\}=f(x),\qquad A_{0x}^{1,\lambda_2}\left\{\frac{d}{dx}B_{0x}^{0,\lambda_2}\left[f(x)\right]\right\}=C_{0x}^{0,\lambda_2}\left[f(x)\right], \tag{11}$$

tengliklarni e’tiborga olsak,

$$\nu(x)=C_{0x}^{0,\lambda_2}\left[\tau(x)\right]-A_{0x}^{1,\lambda_2}\left[\psi'(x/2)\right],\qquad 0<x<1 \tag{12}$$

tenglik hosil bo‘ladi. Bu yerda [3]

$$C_{0x}^{0,\lambda_2}\left[f(x)\right]=signx\left\{\frac{d}{dx}f(x)+\frac{1}{2}\lambda_2^2\int_0^x f(t)\bar{J}_1\left[\lambda_2(x-t)\right]dt\right\} \tag{13}$$

Natijada,

$$\nu(x)=\tau'(x)+\frac{1}{2}\lambda_2^2\int_0^x\tau(t)\bar{J}_1\left[\lambda_2(x-t)\right]dt-\psi_1(x),\qquad 0<x<1 \tag{14}$$

bu yerda

$$\psi_1(x)=A_{0x}^{1,\lambda_2}\left[\psi'(x/2)\right]=\psi'\left(\frac{x}{2}\right)-\int_0^x\psi'\left(\frac{t}{2}\right)\frac{t}{x}\frac{\partial}{\partial t}J_0\left[\lambda_2\sqrt{x(x-t)}\right]dt.$$

(1) tenglama va (2), (3) chegaraviy shartlarda $y\to+0$ limitga o‘tsak,

$$\tau''(x)-\lambda_1^2\tau(x)=\nu(x),\qquad 0<x<1; \tag{15}$$

$$\int_0^1\tau(x)dx=\mu_1(0),\qquad \tau(1)=\mu_2(0) \tag{16}$$

(14) va (8) dan $\nu(x)$ funksiyani chiqarib tashlab, noma’lum $\tau(x)$ funksiyaga nisbatan integral-differensial tenglama hosil qilamiz:

$$\tau''(x)-\tau'(x)-\lambda_1^2\tau(x)-\frac{1}{2}\lambda_2^2\int_0^x\tau(t)\bar{J}_1\left[\lambda_2(x-t)\right]dt=\psi_1(x),\qquad x\in(0,1).$$

Bu yerdan x ni z ga almashtirib, so‘ngra hosil bo‘lgan tenglikni z bo‘yicha $[0,x]$ oraliqda ikki marta integrallash orqali quyidagi ifodani hosil qilamiz:

$$\tau(x)-\int_0^x K(x,t)\tau(t)dt=f(x)+c_1x+c_2, \tag{17}$$

$$f(x)=\int_0^x \psi_1(t)(x-t)dt,$$

bu yerda c_1 va c_2 - ixtiyoriy o‘zgarmas sonlar,

$$K(x,t)=1+\int_t^x\left\{\lambda_1^2+\frac{1}{2}\lambda_2^2\int_t^\xi \bar{J}_1\left[\lambda_2(z-t)\right]dz\right\}d\xi.$$

$\lambda_2\in R$, bo‘lganda $\left|\bar{J}_1[\lambda_2(z-t)]\right|\le 1$ tengsizlik o‘rinlidir. Agar $\lambda_1^2-(1/2)\lambda_2^2\ge 0,$ ya’ni $|\lambda_2|\le\sqrt{2}|\lambda_1|$ bo‘lsa, unda $K(x,t)\ge 1$ o‘rinli bo‘ladi. $K(x,t)\in C^\infty(0\le x,t\le 1)$ agar $\psi(x)\in C^1[0,1/2]\cap C^2(0,1/2)$ bo‘lsa , u holda (17) tenglamaning o‘ng tomoni $C^1[0,1]\cap C^2(0,1)$ shu funksional sinfga tegishli bo‘ladi. Shu sababli Volterra integral tenglamalar nazariyasiga ko‘ra (17) integral tenglamaning yechimi mavjud va u

$$\tau(x)=f(x)+\int_0^x R(x,t)f(t)dt+c_1\left[x+\int_0^x tR(x,t)dt\right]+c_2\left[1+\int_0^x R(x,t)dt\right] \tag{18}$$

ko‘rinishda ifodalanadi. Bu yerda $R(x,t)$ - $K(x,t)$ yadrodan hosil qilingan rezolventa bo‘lib, $R(x,t)\in C^\infty(0\le x,t\le 1)$ va $R(x,t)\ge 1$. (18) funksiyani qanoatlantiruvchi c_1 va c_2 qiymatlari (16) shartdan bir qiymatli aniqlanadi.

$$c_1=\Bigg[c_3\left[\mu_1(0)-\int_0^1 f(x)dx-\int_0^1 dx\int_0^x R(x,t)f(t)dt\right]-c_3c_4\Bigg\{\mu_2(0)-f(1)-\int_0^1 R(1,t)f(t)dt-$$

$$-c_3c_5\left[\mu_1(0)-\int_0^1 f(x)dx-\int_0^1 dx\int_0^x R(x,t)f(t)dt\right]\Bigg\}\Bigg]\left(1+\int_0^1 R(1,t)dt-c_3c_4c_5\right)^{-1}$$

$$c_2=\Bigg[\Bigg\{\mu_2(0)-f(1)-\int_0^1 R(1,t)f(t)dt-c_3c_5\Big[\mu_1(0)-$$

$$-\int_0^1 f(x)dx-\int_0^1 dx\int_0^x R(x,t)f(t)dt\Big]\Bigg\}\left(1+\int_0^1 R(1,t)dt-c_3c_4c_5\right)^{-1}\Bigg]$$

$$c_3=\left(\frac{1}{2}+\int_0^1 dx\int_0^x R(x,t)dt\right)^{-1},\quad c_4=1+\int_0^1 dx\int_0^x R(x,t)dt,\ \ c_5=1+\int_0^1 tR(1,t)dt.$$

Mos ravishda noma’lum $\tau(x)$ funksiya

$$\tau(x)=f(x)+\int_0^x R(x,t)f(t)dt+\left[x+\int_0^x tR(x,t)dt\right]\left[c_3\left[\mu_1(0)-\int_0^1 f(x)dx-\right.\right.$$

$$\left.-\int_0^1 dx\int_0^x R(x,t)f(t)dt\right]-c_3c_4\left\{\mu_2(0)-f(1)-\int_0^1 R(1,t)f(t)dt-c_3c_5\left[\mu_1(0)-\right.\right.$$

$$\left.\left.\left.-\int_0^1 f(x)dx-\int_0^1 dx\int_0^x R(x,t)f(t)dt\right]\right\}\left(1+\int_0^1 R(1,t)dt-c_3c_4c_5\right)^{-1}\right]+$$

$$\left[1+\int_0^x R(x,t)dt\right]\left[\left\{\mu_2(0)-f(1)-\int_0^1 R(1,t)f(t)dt-c_3c_5\left[\mu_1(0)-\right.\right.\right.$$

$$\left.\left.\left.-\int_0^1 f(x)dx-\int_0^1 dx\int_0^x R(x,t)f(t)dt\right]\right\}\left(1+\int_0^1 R(1,t)dt-c_3c_4c_5\right)^{-1}\right]. \qquad (19)$$

ko‘rinishida topiladi.Bundan 1-masala D_1 sohada quyidagi masalaga ekvivalent keltiriladi.

1'-masala. D_1 sohada $u_{xx}-u_y-\lambda_1^2u=0$ tenglama (2), (3) va $u(x,0)=\tau(x)$, $0\le x\le 1$ shartlarni qanoatlantiradi. Bu yerda $\tau(x)$-funksiya (19) tenglik orqali aniqlanadi.

$1'$-masalaning bir qiymatli yechilishini tadqiq qilaylik. Faraz qilaylik, $u(x,y)$ funksiya $1'$-masalaning yechimi hamda $u(0,y)=\mu(y),\ 0\le y\le 1;\ \mu(y)\in C[0,1]$ bo‘lsin. U holda D_1 sohada birinchi chegaraviy masalaning yechimi $u(x,y)$ funksiya[1]

$$u(x,y)=\int_0^1 \tau(\xi)e^{-\lambda_1^2 y}G(x,y;\xi,0)d\xi+\int_0^y \mu(\eta)e^{\lambda_1^2(\eta-y)}G_\xi(x,y;0,\eta)d\eta-$$

$$-\int_0^y \mu_2(\eta)e^{\lambda_1^2(\eta-y)}G_\xi(x,y;1,\eta)d\eta, \qquad (20)$$

ko‘rinishda yoziladi, bu yerda $G(x,y;\xi,\eta)$- Grin funksiyasi

$$G(x,y;\xi,\eta)=\frac{1}{2\sqrt{\pi(y-\eta)}}\sum_{n=-\infty}^{+\infty}\left\{exp\left[-\frac{(x-\xi+2n)^2}{4(y-\eta)}\right]-exp\left[-\frac{(x+\xi+2n)^2}{4(y-\eta)}\right]\right\}.$$

(20) tenglikni x bo‘yicha $[0,1]$ oraliqda inegrallaymiz.

$$\int_0^1 G_\xi(x,y;0,\eta)dx=-\frac{1}{\sqrt{\pi(y-\eta)}}+K_0(y,\eta),$$

tenglikni hisobga olgan holda va $K_0(y,\eta)$ funksiyaning uzluksiz va differensiallanuvchiligidan $\{(y,\eta):0\le\eta<y\le 1\}$ va $\lim\limits_{\eta\to y}K_0(y,\eta)=\lim\limits_{\eta\to y}K_{0y}(y,\eta)=0,$ natijalarni hosil qilamiz.

$$\int_0^1 u(x,y)dx=\int_0^y\left[-\frac{1}{\sqrt{\pi(y-\eta)}}+K_0(y,\eta)\right]e^{\lambda_1^2(\eta-y)}\mu(\eta)d\eta+g_2(y) \qquad (21)$$

bu yerda

$$g_2(y)=\int_0^1\int_0^1\tau(\xi)e^{-\lambda_1^2 y}G(x,y;\xi,0)d\xi dx+\int_0^y\mu_2(\eta)\left\{[\pi(y-\eta)]^{-1/2}-K_0(y,\eta)\right\}e^{\lambda_1^2(\eta-y)}d\eta.$$

(21) ni (2) shartga qo‘yib va $\mu(y)=u(0,y),$ ligidan, $\mu(y)$ ga nisbatan integral tenglamani hosil qilamiz.

$$\frac{1}{\sqrt{\pi}}\int_0^y(y-\eta)^{-1/2}\mu(\eta)d\eta+\int_0^y K_1(y,\eta)\mu(\eta)d\eta=\mu_1(y)-g_2(y), \tag{22}$$

bu yerda

$$K_1(y,\eta)=\left[K_0(y,\eta)e^{\lambda_1^2(\eta-y)}-1\right]+[\pi(y-\eta)]^{-1/2}\left[1+e^{\lambda_1^2(\eta-y)}\right]. \tag{23}$$

$$g_3(y)=\mu_1(y)-g_2(y)-\int_0^y K_1(y,\eta)\mu(\eta)d\eta \tag{24}$$

va $g_3(y)$ funksiyani vaqtincha ma’lum deb hisoblab, (22) tenglamadan $\mu(y)$ ga nisbatan Abel integral tenglamasini hosil qilamiz:

$$\frac{1}{\sqrt{\pi}}\int_0^y(y-\eta)^{-1/2}\mu(\eta)d\eta=g_3(y),\quad 0\le y\le 1. \tag{25}$$

(23) tenglik va (16) shartning biridan foydalanib, $g_3(0)=0,$ ekanligini ko‘rsatish qiyin emas. Ya’ni $g_3(y)$ funksiya (25) tenglama yechimining mavjudligini qanoatlantiradi. (25) tenglamani yechib, $g_3(0)=0,$ ni hisobga olib quyidagini hosil qilamiz:

$$\mu(y)=\frac{1}{\sqrt{\pi}}\int_0^y(y-z)^{-1/2}g_3'(z)dz,\quad 0\le y\le 1.$$

Shu yerdan, (24) belgilashni hisobga olib va ayrim o‘zgarishlarni bajargandan so‘ng ikkinchi turdagi Volterra integral tenglamasi hosil bo‘ladi.

$$\mu(y)-\int_0^y K_2(y,\eta)\mu(\eta)d\eta=g_4(y),\quad 0\le y\le 1, \tag{26}$$

$$g_4(y)=\int_0^y(y-\eta)^{-1/2}[\mu_1'(\eta)-g_2'(\eta)]d\eta,$$

bu yerda

$$K_2(y,\eta)=\frac{1}{\sqrt{\pi}}(y-\eta)^{-1/2}\left\{K_1(\eta,\eta)+\int_\eta^y(y-z)^{-1/2}\frac{\partial}{\partial z}K_1(z,\eta)dz\right\}.$$

Agar $\tau(x)\in C^1[0,1],\ \ \mu_1(y),\mu_2(y)\in C^1[0,1],$ bo‘lsa, u holda $K_2(y,\eta)$ funksiya kuchsiz maxsuslikka ega va $g_4(y)$ funksiya $[0,1]$ uzluksiz bo‘ladi. Shunday qilib, ushbu shartlar bajarilganda (26) tenglama yagona yechimga ega bo‘ladi $\mu(y)\in C[0,1]$. $\mu(y)$ funksiyani topish orqali $u(x,y)$ funksiyani D_1 sohada bir qiymatli aniqlanadi. D_2 sohada masalaning yechimi $u(x,y)$ funksiya (7) formulaga asosan topiladi. $\tau(x)$ funksiya esa mos holda (19) formulaga ko‘ra aniqlanadi.

Shunday qilib, quyidagi natija o‘rinlidir.

Teorema: Faraz qilaylik, agar $|\lambda_2|\le\sqrt{2}|\lambda_1|,\ \ \psi(x)\in C^1[0,1/2]\cap C^2(0,1/2),\ \ \mu_1(y),$

$\mu_2(y) \in C^1[0,1],$ bo‘lsa 1-masal yagona yechimga ega.

Xulosa

Ushbu maqola ikkinchi tartibli spektral parametrli parabolo-giperbolik tipdagi tenglama uchun integral shartli masalani bayon qilish va tadqiq etishga bag‘ishlangan. Qo‘yilgan masalalarni D_2 sohada tadqiq etishda integro-differensial opertorlar va ularning xossalaridan foydalanib, noma’lum $\tau\left(x\right)$ va $\nu\left(x\right)$ funksiyalar o‘rtasidagi munosabat olingan.

O‘rganilgan masalani D_1 sohada tadqiq qilishda esa spektral parametrli ikkinchi tartibli parabolik tipdagi tenglama uchun birinchi chegaraviy masala yechimi ko‘rinishi va integral tenglamalar nazariyasidan foydalanilgan.

Maqolada tenglamada spektral parametrni qo‘shilishi masalalarni tadqiq qilish usuliga ta’sir etishi yuzasidan xulosa qilindi.

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